Time decay for solutions to the Stokes equations with drift

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Abstract

In this note, we study the behaviour of Lebesgue norms $\|v(\cdot,t)\|_p$ of solutions v to the Cauchy problem for the Stokes system with drift u, which is supposed to be a divergence free smooth vector valued function satisfying a scale invariant condition.

1 Inroduction

The main aim of the paper is the following Stokes system with a drift u

$$\partial_t v - u\cdot\nabla v - \triangle v - \nabla q = -\mathrm{div}\, F, \qquad \mathrm{div}\, v = 0 \tag{1.1}$$

in $Q_+ = \mathbb{R}^3\times]0,\infty[$ and

$$v(x,0) = 0 \tag{1.2}$$

for $x\in\mathbb{R}^3$.

It is supposed that a tensor-valued field F is smooth and compactly supported in Q_+. In addition, let us assume that F is skew symmetric and therefore

$$\mathrm{div}\,\mathrm{div}\, F = 0. \tag{1.3}$$

As to the drift u, one may assume that u is a bounded divergence free field in Q_+, say $|u|\leq 1$ there, whose derivatives of any order exist and are bounded in Q_+.

It is not so difficult to prove, see Appendix I, the following statement.

Proposition 1.1. *There exists a unique solution v to (1.1) and (1.2) with properties:*

$$\nabla^l\partial_t^k v\in L_2(Q_+)$$

for $k,l=0,1,...$ *except* $k+l=0$,

$$\nabla^{l+1}\partial_t^k q\in L_2(Q_+)$$

for $k,l=0,1,...$,

$$v\in L_{2,\infty}(Q_+),\qquad q\in L_{2,\infty}(Q_+)$$

for any $k=0,1,....$

The goal of the paper is to study how L_p-norms of the velocity field v $\Big(\|v(\cdot,t)\|_p:=\Big(\int\limits_{\mathbb{R}^3}|v(x,t)|^p dx\Big)^{\frac{1}{p}}\Big)$ behave as $t\to\infty$. In particular, two cases are of great interest: $p=1$ and $p=2$.

Let us impose a decay assumption on the drift

$$|u(x,t)|\leq\frac{c_d}{|x|+\sqrt{t}}\tag{1.4}$$

for all $(x,t)\in Q_+$.

Two results will be proven in the paper.

Theorem 1.2. *Let* v *be a solution* v *to (1.1) and (1.2) and let* u *satisfy (1.4). Then for any* $m=0,1...$, *two decay estimates are valid:*

$$\|v(\cdot,t)\|_1\leq c(m,c_d)\sqrt{t}^{\frac{3}{2}}\frac{1}{\ln^m(t+e)}\tag{1.5}$$

and

$$\|v(\cdot,t)\|_2\leq\frac{c(m,c_d)}{\ln^m(t+e)}.\tag{1.6}$$

To motivate the aforesaid problem and the assumptions made, consider the Navier-Stokes system

$$\partial_t w+w\cdot\nabla w-\Delta w=-\nabla r,\qquad \operatorname{div}w=0$$

in the unit parabolic ball $Q=B\times]-1,0[$ for functions $w\in L_\infty(-1,0;L_2(B))\cap L_2(-1,0;W^1_2(B))$ and $r\in L_{\frac{3}{2}}(Q)$ satisfying the additional restriction

$$|w(x,t)|\leq\frac{c_d}{|x|+\sqrt{-t}}\tag{1.7}$$

for all $(x,t) \in Q$. Our aim is to understand whether or not the origin $z = (x,t) = (0,0)$ is a regular point of w, i.e., there exists $\delta > 0$ such that v is essentially bounded in the parabolic ball $Q(\delta) = B(\delta)\times]-\delta^2, 0[$. Here, as usual, $B(r)$ stands for the ball of radius r centered at the origin. The answer is certainly positive if c_d is sufficiently small. However, we would not like to make such an assumption at this point. In [8], it has been shown that if $z = 0$ is a singular point of w then a so-called a mild bounded ancient solution $\tilde{u}$ to the Navier-Stokes equations in $Q_- = \mathbb{R}^3\times]-\infty, 0[$ exists and it is non-trivial. The latter means the following: $\tilde{u} \in L_\infty(Q_-)$ ($|\tilde{u}| \leq 1$ a.e. in $Q_-$and $|u(0)| = 1$) and there exists a scalar function $\tilde{p} \in L_\infty(-\infty, 0; BMO(\mathbb{R}^3))$ such that the pair $\tilde{u}$ and $\tilde{p}$ satisfy the classical Navier-Stokes system

$$\partial_t\tilde{u} + \tilde{u}\cdot\nabla\tilde{u} - \triangle\tilde{u} = -\nabla\tilde{p}, \qquad \operatorname{div}\tilde{u} = 0 \tag{1.8}$$

in Q_- in the sense of distributions. It is known, see [4], that $\tilde{u}$ is infinitely smooth and all its derivatives are bounded. Moreover, it can be shown, see Appendix II, that, for $u(x,t) = \tilde{u}(x,-t)$,

$$\int\limits_{Q_+} u\cdot\operatorname{div} F dxdt = -\lim_{T\to\infty}\int\limits_{\mathbb{R}^3} u(x,T)\cdot v(x,T)dx. \tag{1.9}$$

If time decay of v is such that, for any tensor-valued field $F \in C_0^\infty(\mathbb{R}^3)$, obeying condition (1.3), the limit on the right hand side of (1.9) vanishes, then one can easily show that u must be a function of time only. Indeed, we then have

$$\int\limits_{Q_+} \nabla u : F dxdt = 0.$$

The latter means that the skew symmetric part of ∇u vanishes in Q_+. Since u is a divergence free field, u is a bounded harmonic function and so does $\tilde{u}$ in Q_-. But $\tilde{u}$ is a bounded mild ancient solution to the Navier-Stokes equation and thus must be a constant in Q_- as well as u in Q_+. But condition (1.4) means that $\tilde{u}$ is identically zero. This finally would prove that $z = 0$ is not a singular point of w and condition (1.7) is in fact a regularity condition.

Unfortunately, decay bounds in Theorem 1.2 do not provide the above scenario. Let us give a couple of bounds on c_d that give a required time decay.

To describe the first case, we are going to use a solution formula for the Stokes system with non-divergence free right hand side.

Let

$$\mathcal{F} = -v \otimes u + F.$$

The solution to problem (1.1), (1.2) has the form, see for instance [4],

$$v(x,t) = \int_0^t \int_{\mathbb{R}^3} K(x-y, t-s)\mathcal{F}(y,s)dyds, \tag{1.10}$$

where the potential $K = (K_{ijl})$ defined with the help of the standard heat kernel in the following way

$$\Delta\Phi(x,t) = \Gamma(x,t)$$

and

$$K_{ijl} = \Phi_{,ijl} - \delta_{il}\Phi_{,kkj}.$$

It is easy to check that the following bound is valid:

$$|K(x,t)| \leq \frac{c_1}{(t+|x|^2)^2} \tag{1.11}$$

and therefore

$$\int_{\mathbb{R}^3} |K(x,t)|dx \leq \frac{c_*}{\sqrt{t}} \tag{1.12}$$

with $c_* = cc_1$, where c is an absolute constant.

Theorem 1.3. *Assume that*

$$4c_*c_d < 1. \tag{1.13}$$

Then

$$\int_{\mathbb{R}^3} v(x,T) \cdot u(x,T)dx \to 0 \tag{1.14}$$

as $T \to \infty$.

To descibe the second case, let us introduce the operator $K : \mathcal{L}_2 \to J_2$, where $\mathcal{L}_2$ consists of all tensor-valued functions, belonging to $L_2(\mathbb{R}^3)$ and satisfying condition (1.3), and J_2 is a space of square integrable divergence free fields in $\mathbb{R}^3$. The action of this operator is defined as $A_F = KF$, where A_F is the unique solution to the following problem

$$\operatorname{rot} A_F = -\operatorname{div} F. \tag{1.15}$$

The elliptic theory reads that operator K is bounded.

In addition, one may introduce the second operator $M : L_2(\mathbb{R}^3; \mathbb{M}^{3\times 3}) \to L_2(\mathbb{R}^3)$ so that

$$\Delta q_F = -\operatorname{div}\operatorname{div} F, \tag{1.16}$$

where $q_F = MF$.

Actually, we have fixed the pressure $q = q_{v\otimes u}$ in Proposition 1.1. This will be done everywhere in what follows. Our result is the following.

Theorem 1.4. *Let*

$$c_d \leq \frac{\sqrt{3}}{2\|K\|(1+\sqrt{3}\|M\|)}.$$

Then (1.14) is true.

2 Time Decay of L_1-Norm

Now, from (1.10), it follows

$$\|v(\cdot,t)\|_p \leq \int\limits_0^t \| \int\limits_{\mathbb{R}^3} |K(\cdot - y, t-s)|\mathcal{F}(y,s)dy\|_p ds$$

Applying Hölder inequality and taking into account (1.12), we find

$$\|v(\cdot,t)\|_p \leq \int\limits_0^t \Big(\int\limits_{\mathbb{R}^3} \Big(\int\limits_{\mathbb{R}^3} K(x-y', t-s)dy'\Big)^{\frac{p}{p'}} \times$$

$$\times \int\limits_{\mathbb{R}^3} |K(x-y,t-s)||\mathcal{F}(y,s)|^p dydx\Big)^{\frac{1}{p}} ds \leq c\int\limits_0^t \frac{1}{\sqrt{t-s}}\|\mathcal{F}(\cdot,s)\|_p ds$$

for any $p \geq 1$.

Now, for p, satisfying the condition

$$p \in]6/5, 2[, \tag{2.1}$$

Hölder inequality gives the following estimate

$$\|v(\cdot,t)\|_p \le c\int\limits_0^t \frac{ds}{\sqrt{t-s}}\Big(\int\limits_{\mathbb{R}^3}|\mathcal{F}(y,s)|^2(\sqrt{s}+|y|)^2dy\Big)^{\frac{1}{2}}\times$$

$$\times\Big(\int\limits_{\mathbb{R}^3}\Big(\frac{1}{\sqrt{s}+|y|}\Big)^{\frac{2p}{2-p}}dy\Big)^{\frac{2-p}{2p}}.$$

By changing variables $y=z\sqrt{s}$,

$$\Big(\int\limits_{\mathbb{R}^3}\Big(\frac{1}{\sqrt{s}+|y|}\Big)^{\frac{2p}{2-p}}dy\Big)^{\frac{2-p}{2p}} \le \sqrt{s}^{-\frac{5p-6}{2}}\Big(\int\limits_{\mathbb{R}^3}\Big(\frac{1}{1+|z|}\Big)^{\frac{2p}{2-p}}dz\Big)^{\frac{2-p}{2p}} =$$

$$= C(p)\sqrt{s}^{-\frac{5p-6}{2p}}$$

with

$$C(p):=\Big(\int\limits_{\mathbb{R}^3}\Big(\frac{1}{1+|z|}\Big)^{\frac{2p}{2-p}}dz\Big)^{\frac{2-p}{2p}}\to\infty$$

as $p\to 6/5+0$. So,

$$\|v(\cdot,t)\|_p \le C(p)\int\limits_0^t \frac{ds}{\sqrt{t-s}}\sqrt{s}^{-\frac{5p-6}{2p}}\Big(\int\limits_{\mathbb{R}^3}|\mathcal{F}(y,s)|^2(\sqrt{s}+|y|)^2dy\Big)^{\frac{1}{2}}.$$

Now, by our assumptions on F and by (1.4),

$$\int\limits_{\mathbb{R}^3}|\mathcal{F}(y,s)|^2(\sqrt{s}+|y|)^2dy \le c(c_d\|v(\cdot,s)\|_2+\|G(\cdot,s)\|_2)^2, \qquad (2.2)$$

where $G(y,s)=F(y,s)(\sqrt{s}+|y|)$, and thus

$$\|v(\cdot,t)\|_p \le C(p)\int\limits_0^t \frac{ds}{\sqrt{t-s}}\sqrt{s}^{-\frac{5p-6}{2p}}(c_d\|v(\cdot,s)\|_2+\|G(\cdot,s)\|_2)$$

$$\le C(p)A_p(s)$$

with

$$A_p(s) := \int\limits_0^t \frac{ds}{\sqrt{t-s}} \sqrt{s}^{-\frac{5p-6}{2p}} (c_d\|v(\cdot,s)\|_2 + \|G(\cdot,s)\|_2). \tag{2.3}$$

So,

$$\|v(\cdot,t)\|_p \leq C(p)A_p(t). \tag{2.4}$$

Now, one can repeat the above arguments for $p=1$ and find

$$\|v(\cdot,t)\|_1 \leq \int\limits_0^t \frac{c}{\sqrt{t-s}} \int\limits_{\mathbb{R}^3} |\mathcal{F}(y,s)| dyds.$$

Since

$$|\mathcal{F}(y,s)| \leq c\frac{c_d|v(y,s)| + |G(y,s)|}{\sqrt{s}+|y|},$$

the latter estimate can be transform as follows:

$$\|v(\cdot,t)\|_1 \leq c\int\limits_0^t \frac{ds}{\sqrt{t-s}} \int\limits_{\mathbb{R}^3} \frac{c_d|v(y,s)| + |G(y,s)|}{\sqrt{s}+|y|} dy \leq$$

$$\leq c\int\limits_0^t \frac{ds}{\sqrt{t-s}} \Big(\int\limits_{\mathbb{R}^3} \Big(\frac{1}{\sqrt{s}+|y|}\Big)^{\frac{6+5\varepsilon}{1+5\varepsilon}} dy\Big)^{\frac{1+5\varepsilon}{6+5\varepsilon}} \Big(\int\limits_{\mathbb{R}^3} (c_d|v(y,s)| +$$

$$+|G(y,s)|)^{\frac{6+5\varepsilon}{5}} dy\Big)^{\frac{5}{6+5\varepsilon}}$$

for some positive $0<\varepsilon<3/10$. Hence,

$$\|v(\cdot,t)\|_1 \leq C_1(\varepsilon) \int\limits_0^t \frac{ds}{\sqrt{t-s}} \sqrt{s}^{3\frac{1+5\varepsilon}{6+5\varepsilon}-1} \Big(\int\limits_{\mathbb{R}^3} (c_d|v(y,s)| +$$

$$+|G(y,s)|)^{\frac{6+5\varepsilon}{5}} dy\Big)^{\frac{5}{6+5\varepsilon}}$$

with

$$C_1(\varepsilon) := \Big(\int\limits_{\mathbb{R}^3} \Big(\frac{1}{1+|z|}\Big)^{\frac{6+5\varepsilon}{1+5\varepsilon}} dz\Big)^{\frac{1+5\varepsilon}{6+5\varepsilon}}.$$

Simplifying slightly the previous bound, we have

$$\|v(\cdot,t)\|_1 \leq C_1(\varepsilon) \int\limits_0^t \frac{ds}{\sqrt{t-s}} \sqrt{s}^{\frac{-3+10\varepsilon}{6+5\varepsilon}} (\|c_d v(\cdot,s)\|_{\frac{6+5\varepsilon}{5}} + \|G(\cdot,s)\|_{\frac{6+5\varepsilon}{5}}) dy.$$

By (2.4),

$$\|v(\cdot,s)\|_{\frac{6+5\varepsilon}{5}} \leq C(6/5+\varepsilon) A_{\frac{6}{5}+\varepsilon}(t).$$

So, the final estimate of L_1-norm is:

$$\|v(\cdot,t)\|_1 \leq C_3(\varepsilon) \int\limits_0^t \frac{ds}{\sqrt{t-s}} \sqrt{s}^{\frac{-3+10\varepsilon}{6+5\varepsilon}} (c_d A_{\frac{6}{5}+\varepsilon}(s) +$$

$$+ \|G(\cdot,s)\|_{\frac{6+5\varepsilon}{5}}) dy \tag{2.5}$$

with $C_3(\varepsilon) \to \infty$ as $\varepsilon \to 0$.

Since the energy of v is bounded, one can derive from (2.3) the following:

$$A_p(t) \leq c(s)(c_d\|v\|_{2,\infty} + \|G\|_{2,\infty}) \sqrt{t}^{\frac{6-3p}{2p}}$$

and thus

$$A_{\frac{6}{5}+\varepsilon}(t) \leq c(\varepsilon)(c_d\|v\|_{2,\infty} + \|G\|_{2,\infty}) \sqrt{t}^{\frac{12-15\varepsilon}{2(6+5\varepsilon)}}.$$

Now, (2.5) is giving to us:

$$\|v(\cdot,t)\|_1 \leq c\sqrt{t}^{\frac{3}{2}}$$

where c depends on the data of the problem.

3 Improvement for L_2-norm

Here, we are going to use methods developed in [5] and [11], see also [1] and [3].

We have the energy inequality

$$\partial_t y(t) + \|\nabla v(\cdot,t)\|_2^2 \leq \|F(\cdot,t)\|_2^2 \tag{3.1}$$

with $y(t) = \|v(\cdot,t)\|_2^2$.

The Fourier transform and Plancherel identity give us

$$\partial_t y(t) \le - \int\limits_{\mathbb{R}^3} |\xi|^2 |\widehat{v}(\xi,t)|^2 d\xi + \|F(\cdot,t\|_2^2 \le$$

$$= - \int\limits_{|\xi|>g(t)} |\xi|^2 |\widehat{v}(\xi,t)|^2 d\xi - \int\limits_{|\xi|\le g(t)} |\xi|^2 |\widehat{v}(\xi,t)|^2 d\xi + \|F(\cdot,t\|_2^2,$$

where $g(t)$ is a given function which will be specified later on. The latter implies

$$y'(t) + g^2(t) y(t) \le \int\limits_{|\xi|\le g(t)} (g^2(t) - |\xi|^2) |\widehat{v}(\xi,t)|^2 d\xi + \|F(\cdot,t\|_2^2.$$

Taking the Fourier transform of the Navier-Stokes equation, we find

$$\partial_t \widehat{v} + |\xi|^2 \widehat{v} = -\widehat{H},$$

where

$$H = -\mathrm{div}\,(v \otimes u + \mathbb{I} q - F).$$

Clearly,

$$\widehat{v}(\xi,t) = -\int_0^t \exp\{-|\xi|^2(t-s)\} \widehat{H}(\xi,s) ds$$

and

$$|\widehat{H}(\xi,s)| \le |\xi| \Big(\||v(\cdot,s)||u(\cdot,s)|\|_1 + \|F(\cdot,s)\|_1 \Big).$$

Denoting

$$k(t) = \||v(\cdot,t)|\|_1,$$

we notice

$$\||v(\cdot,s)||u(\cdot,s)|\|_1 \le \sqrt{s}^{-1} c_d k(s).$$

So,

$$|\widehat{v}(\xi,t)| \le c \int_0^t \exp\{-|\xi|^2(t-s)\} |\xi| (\sqrt{s}^{-1} k(s) + \|F(\cdot,s)\|_1) ds.$$

Applying the Hölder inequality, we show

$$y'(t) + g^2(t) y(t) \le$$

$$\leq c \int\limits_{|\xi|\leq g(t)} (g^2(t)-|\xi|^2)\Big(\int_0^t \exp\{-|\xi|^2(t-s)\}|\xi|(\sqrt{s}^{-1}k(s)+\|F(\cdot,s\|_1)ds\Big)^2 \leq$$

$$\leq c\int_0^t (s^{-1}k^2(s)+\|F(\cdot,s\|_1^2)ds\times$$

$$\times \int_0^t \int\limits_{|\xi|\leq g(t)} (g^2(t)-|\xi|^2)|\xi|^2 \exp\{-2|\xi|^2(t-s_1)\}d\xi ds_1 + \|F(\cdot,t)\|_2^2.$$

The latter integral can be estimated in the following way:

$$\int\limits_0^t \int\limits_{|\xi|\leq g(t)} (g^2(t)-|\xi|^2)|\xi|^2 \exp\{-|\xi|^2(t-s_1)\}d\xi ds_1 =$$

$$= c\int\limits_0^t \int\limits_0^{g(t)} (g^2(t)-r^2)r^4 \exp\{-2r^2(t-s_1)\}dr ds_1 \leq$$

$$\leq cg^6(t)\int\limits_0^t \int\limits_0^{g(t)} \exp\{-2r^2(t-s_1)\}d(r\sqrt{t-s_1})\frac{ds_1}{\sqrt{t-s_1}} \leq$$

$$\leq cg^6(t)\int\limits_0^t \frac{ds_1}{\sqrt{t-s_1}}\int\limits_0^\infty \exp\{-2z^2\}dz \leq cg^6(t)\sqrt{t}.$$

Coming back to our energy inequality, we find

$$y'(t)+g^2(t)y(t)\leq$$

$$\leq K(t) := cg^6(t)\sqrt{t}\int_0^t (s^{-1}k^2(s)+\|F(\cdot,s\|_1^2)ds + \|F(\cdot,t)\|_2^2.$$

Then Grownwall inequality implies

$$y(t)\leq c\int\limits_0^t \exp\Big\{-\int\limits_s^t g^2(\tau)d\tau\Big\}K(s)ds.$$

4 Proof of Theorem 1.2

The proof is on induction in m. The basis of induction has been already established in Section II. Let us assume that our statement is true for m and show that it is true for $m+1$.

We can estimate $K(t)$ using the fact that F has a compact support

$$K(t) \leq cg^6(t)\sqrt{t}\int_0^t (\sqrt{s}\ln^{-2m}(s+e) + \|F(\cdot,s)\|_1^2)ds + \|F(\cdot,t)\|_2^2 \leq$$

$$\leq C(\|F\|_{1,\infty}, m)g^6(t)\sqrt{t}\int_0^t \sqrt{s}\ln^{-2m}(s+e)ds + \|F(\cdot,t)\|_2^2 \leq$$

$$\leq C(\|F\|_{1,\infty}, m)g^6(t)t^2\ln^{-2m}(t+e) + \|F(\cdot,t)\|_2^2.$$

Let

$$g^2(t) = \frac{h'(t)}{h(t)}. \tag{4.1}$$

Then

$$\int\limits_0^t \exp\Big\{-\int\limits_s^t g^2(\tau)d\tau\Big\}(g^6(s)s^2\ln^{-2m}(s+e) + \|F(\cdot,s)\|_2^2)ds =$$

$$= \frac{1}{h(t)}\int\limits_0^t \Big(\frac{s^2\ln^{-2m}(s+e)}{h^2(s)}(h'(s))^3 + h(s)\|F(\cdot,s)\|_2^2\Big)ds.$$

Now, one specify function g by a particular choice of fuction h, setting

$$h(t) = \ln^k(t+e) \tag{4.2}$$

for some $k > 2m+2$. Then

$$\frac{1}{h(t)}\int\limits_0^t \frac{s^2\ln^{-2m}(s+e)}{h^2(s)}(h'(s))^3 ds =$$

$$= \frac{1}{\ln^k(t+e)}\int\limits_0^t \frac{s^2\ln^{-2m}(s+e)}{(s+e)^3}k^3\ln^{k-3}(s+e)ds \leq$$

$$= \frac{1}{\ln^k(t+e)} \int_0^t \frac{s^2}{(s+e)^3} k^3 \ln^{k-2m-3}(s+e)ds \le$$

$$\le \frac{1}{\ln^k(t+e)} \frac{k^3}{k-2m-2} (\ln^{k-2m-2}(t+e) - 1) \le$$

$$\le c(k,m) \frac{1}{\ln^{2m+2}(t+e)}.$$

Since $s \mapsto \|F(\cdot,s)\|_2^2$ has a compact support in $]0,\infty[$, we find

$$\|v(\cdot,t)\|_2 \le \frac{c}{\ln^{m+1}(t+e)}.$$

Then, as it follows from (2.3),

$$A_p(t) \le C(\|G\|_{2,\infty}, p, m) \sqrt{t}^{\frac{6-3p}{2p}} \frac{1}{\ln^{m+1}(t+e)}$$

and thus

$$A_{\frac{6}{5}+\varepsilon}(t) \le C(\|G\|_{2,\infty}, \varepsilon, m) \sqrt{t}^{\frac{12-15\varepsilon}{2(6+5\varepsilon)}} \frac{1}{\ln^{m+1}(t+e)}.$$

And again from (2.5), it follows finally that

$$\|v(\cdot,t)\|_1 \le c\sqrt{t}^{\frac{3}{2}} \frac{1}{\ln^{m+1}(t+e)}.$$

5 Liouville type theorems

Proof of Theorem 1.3 From (1.10) and from (1.4), one can derive

$$f(t) \le c_* \int_0^t \frac{1}{\sqrt{t-s}} \Big(\frac{c_d}{\sqrt{s}} f(s) + \|F(\cdot,s)\|_1 \Big) ds, \tag{5.1}$$

where $f(t) := \|v(\cdot,t)\|_1$. Since F is compactly supported, (5.1) can be reduced to the following form:

$$f(t) \le A + c_* c_d \int_0^t \frac{1}{\sqrt{t-s}\sqrt{s}} f(s)ds.$$

Now, fix an arbitrary $T > 0$. Then, for any $t \in]0, T]$, we have

$$f(t) \leq A + 4c_* c_d M(T),$$

where $M(T) = \sup\limits_{0<t\leq T} f(t)$. Hence,

$$M(T) \leq A + 4c_* c_d M(T)$$

for any $T > 0$. Finally, we see that

$$\|v(\cdot, t)\|_1 \leq c = \frac{A}{1 - 4c_* c_d}$$

for all $t > 0$. Therefore,

$$\Big|\int\limits_{\mathbb{R}^3} u(x,t) \cdot v(x,t) dx\Big| \leq \frac{c}{\sqrt{t}} \to 0$$

as $t \to \infty$. ✷

Proof of Theorem 1.4 Assume that F is skew symmetric and therefore satisfies condition (1.3).

Equation (1.1) can be written as follows:

$$\partial_t v - \Delta v = \operatorname{div} F_0, \tag{5.2}$$

where

$$F_0 = v \otimes u + \nabla q \mathbb{I} - F.$$

We know from previous results that

$$F_0 \in L_{2,\infty}(Q_+), \qquad \operatorname{div} F_0 \in L_2(Q_+). \tag{5.3}$$

Since $\operatorname{div} \operatorname{div} F_0 = 0$, we can apply the elliptic theory and conclude that there exists a divergence free field $A(\cdot, t)$ such that

$$\operatorname{rot} A(\cdot, t) = \operatorname{div} F_0(\cdot, t) \tag{5.4}$$

in $\mathbb{R}^3$ and the following estimate holds

$$\|A(\cdot, t)\|_2 \leq \|K\| \|F_0(\cdot, t)\|_2 \tag{5.5}$$

for all $t \in]0,\infty[$. Taking into account the definition of the operator M, one can go further and derive from (5.5)

$$\|A(\cdot,t)\|_2 \leq \|K\|(\|v(\cdot,t)\otimes u(\cdot,t)\|_2 + \|q_{v\otimes u}(\cdot,t)\mathbb{I}\|_2 + \|F(\cdot,t)\|_2) \leq$$
$$\leq \|K\|(1+\sqrt{3}\|M\|)\|v(\cdot,t)\otimes u(\cdot,t)\|_2 + h(t),$$

where $h(t) = \|K\|\|F(\cdot,t)\|_2$ and thus

$$\|A(\cdot,t)\|_2 \leq \|K\|(1+\sqrt{3}\|M\|)\frac{c_d}{\sqrt{t}}\|v(\cdot,t)\|_2 + h(t) \tag{5.6}$$

With the above A, let us consider the Cauchy problem

$$\partial_t B - \Delta B = A \tag{5.7}$$

$$B(\cdot,0) = 0. \tag{5.8}$$

Problem (5.7), (5.8) has a unique solution defined for all positive t and $B \in W^{2,1}_2(Q_T)$ for all $T > 0$. Since $A(\cdot,t)$ is divergence free, so is $B(\cdot,t)$. Now, let $w = \operatorname{rot} B$. Then we can see that w is a solution to equation (5.2) and sinse it vanishes at $t = 0$, we can state that $w = v$.

Now, let us analyse the Cauchy problem for B. It is easy to see that B satisfies the energy identity

$$\frac{1}{2}\partial_t\|B(\cdot,t)\|_2^2 + \|\nabla B(\cdot,t)\|_2^2 = \int\limits_{\mathbb{R}^3} A(x,t)\cdot B(x,t)dx. \tag{5.9}$$

Taking into account the simple identity

$$\|v(\cdot,t)\|_2 = \|\nabla B(\cdot,t)\|_2,$$

one can derive from (5.6) the following estimate

$$\frac{1}{2}\partial_t\|B(\cdot,t)\|_2^2 + \|v(\cdot,t)\|_2^2 \leq \|K\|(1+\sqrt{3}\|M\|)\frac{c_d}{\sqrt{t}}\|v(\cdot,t)\|_2\|B(\cdot,t)\|_2+$$
$$+h(t)\|B(\cdot,t)\|_2.$$

Applying the Young inequality, we find

$$\frac{1}{2}\partial_t\|B(\cdot,t)\|_2^2 \leq \|K\|^2(1+\sqrt{3}\|M\|)^2\frac{c_d^2}{4t}\|B(\cdot,t)\|_2^2 + \frac{1}{2}h(t)(\|B(\cdot,t)\|_2^2+1)$$

Let us introduce the important constant

$$l = \|K\|^2(1+\sqrt{3}\|M\|)^2\frac{c_d^2}{2}.$$

Then the previous inequality leads to

$$\|B(\cdot,t)\|_2^2 \le t^l\int\limits_0^t \frac{h(\tau)}{\tau^l}\exp\Big(-\int_\tau^t h(s)ds\Big)d\tau.$$

Taking into account that F is compactly supported in Q_+, we have

$$\|B(\cdot,t)\|_2^2 \le c_F t^l.$$

From here, it is easy to derive the following:

$$\int\limits_o^t \|v(\cdot,s)\|_2^2 ds \le c_F t^l. \tag{5.10}$$

We denote all the constant depending of F and its support by c_F.

Having estimate (5.10) in mind, let us go back to equation (5.2) multiplying it by tv and integrating result over $\mathbb{R}^3\times]0,t[$, as a result, we find the following differential inequality

$$\frac{1}{2}t\|v(\cdot,t)\|_2^2+\int\limits_0^t\|\nabla v(\cdot,s)\|_2^2ds=\frac{1}{2}\|v(\cdot,t)\|_2^2+\int\limits_0^t\int\limits_{\mathbb{R}^3} sF(x,s)\cdot v(x,s)ds \le$$

$$\le c_F(\int\limits_0^t \|v(\cdot,s)\|_2^2ds+1).$$

The latter, together with boundedness of $\|v(\cdot,t)\|_2$, implies the bound

$$\|v(\cdot,t)\|_2^2 \le c_F(t+1)^{l-1},$$

which, in turn, allows to improve the decay of $\|v(\cdot,t)\|_1$. To this end, we are going back to (2.4) and (2.5). Indeed, by the assumption of the theorem $l<3/4$,

$$A_p(t)\le c\int\limits_0^t\frac{1}{\sqrt{t-s}}s^{-\frac{5p-6}{4p}}(s+1)^{l-1}ds \le c\int\limits_0^t\frac{1}{\sqrt{t-s}}s^{-\frac{5p-6}{4p}+l-1}ds \le$$

$$\leq ct^{\frac{6-3p}{4p}+l-1}.$$

Letting $p = 6/5 + \varepsilon$, for sufficiently small positive ε, we find

$$\|v(\cdot,t)\|_1 \leq c(\sqrt{t})^{\frac{3}{2}+2(l-1)}.$$

This shows

$$\Big|\int\limits_{\mathbb{R}^3} v(\cdot,t)\cdot u(\cdot,t)dx\Big| \leq c(\sqrt{t})^{\frac{1}{2}+2(l-1)} \to 0$$

as $t \to \infty$ provided $l < \frac{3}{4}$. ✷

6 Appendix I

Proof We recall that all derivatives of u are bounded.

First of all, there exists a unique energy solution. This follows from the identity

$$\int\limits_{Q_+} (u\cdot\nabla v)\cdot v dxdt = 0$$

and from the inequality

$$\Big| -\int\limits_{Q_+} \operatorname{div} F\cdot v dxdt\Big| = \Big|\int\limits_{Q_+} F:\nabla v dxdt\Big| \leq \Big(\int\limits_{Q_+}|F|^2dxdt\Big)^{\frac{1}{2}}\Big(\int\limits_{Q_+}|\nabla v|^2dxdt\Big)^{\frac{1}{2}}$$

So, we can state that

$$v\in L_{2,\infty}(Q_+), \qquad \nabla v\in L_2(Q_+). \tag{6.1}$$

The latter means that $u\cdot\nabla v\in L_2(Q_+)$. The pressure can be recovered from the pressure equation

$$\triangle q = \operatorname{div}\operatorname{div}(F - v\otimes u).$$

One of solutions to the above equation has the form

$$q_0(x,t) = -\frac{1}{3}v(x,t)\cdot u(x,t) + \lim_{\varepsilon\to 0}\int\limits_{|x-y|>\varepsilon} \nabla^2 E(x-y) : v(y,t)\otimes u(y,t)dy,$$

where E is the fundamental solution to the Laplace operator. All others differs from q_0 by a function of time only. Let us fix the pressure by setting $q = q_0$. The theory of singular integrals implies that

$$q \in L_{2,\infty}(Q_+), \qquad \nabla q \in L_2(Q_+).$$

Then, by properties of solutions to the heat equation, we have

$$\nabla^2 v \in L_2(Q_+) \qquad \partial_t v \in L_2(Q_+).$$

Going back to the pressure equation, let us re-write it in the following way

$$\triangle q = \operatorname{div}\operatorname{div} F - u_{i,j}v_{j,i} \in L_2(Q_+)$$

and thus

$$\nabla^2 q \in L_2(Q_+).$$

Next, since u is infinitely smooth and all its derivatives are bounded in space and time, after differentiation with respect to x_k, we find

$$\partial_t v_{,k} - \triangle v_{,k} = \nabla q_{,k} - \operatorname{div} F_{,k} + u_{,k} \cdot \nabla v + u \cdot \nabla v_{,k} \in L_2(Q_+)$$

and therefore

$$\partial_t \nabla v,\ \nabla^3 v \in L_2(Q_+).$$

Arguing in the same way, we find

$$\partial_t \nabla^k v,\ \nabla^{k+2} v,\ \nabla^{k+1} q \in L_2(Q_+)$$

for each $k = 0, 1, ...$

Now, we differentiate in t the pressure equation

$$\triangle \partial_t q = \operatorname{div}\left(\operatorname{div} \partial_t F - \partial_t u \cdot \nabla v - u \cdot \nabla \partial_t v\right)$$

and establish

$$\nabla^k \partial_t q \in L_2(Q_+)$$

for any $k = 1, 2, ...$ Then

$$\partial_t^2 v - \triangle \partial_t v = -\operatorname{div} \partial_t F + \nabla \partial_t q + \partial_t u \cdot \nabla v + u \cdot \nabla \partial_t v$$

and thus

$$\nabla^k \partial_t^2 v,\ \nabla^{k+2} \partial_t v \in L_2(Q_+)$$

for $k = 0, 1, ...$ And so on. ✷

7 Appendix II

We recall that $u(x,t)=\tilde{u}(x,-t)$ and $p(x,t)=-\tilde{p}(x,-t)$ for $t>0$. Then

$$-\partial_t u+u\cdot\nabla u-\triangle u=-\nabla p,\qquad \mathrm{div}\,u=0 \tag{7.1}$$

in Q_+ in the sense of distributions.

So, let v be a solution to (1.1) and (1.2). Now, for a compactly supported smooth function ψ in Q_+, integration by parts gives

$$\int\limits_{Q_+}u\cdot\psi\mathrm{div}\,Fdxdt=$$

$$=\int\limits_{Q_+}u\cdot\psi\Big(-\partial_t v+u\cdot\nabla v+\triangle v+\nabla q\Big)dxdt=$$

$$=\int\limits_{Q_+}\Big(u\cdot v\partial_t\psi-u\cdot vu\cdot\nabla\psi-u_iv_{i,j}\psi_{,j}+u_{i,j}v_i\psi_{,j}-qu\cdot\nabla\psi\Big)dxdt+$$

$$+v\psi\cdot\Big(\partial_t u-u\cdot\nabla u+\triangle u\Big)dxdt=$$

$$=\int\limits_{Q_+}\Big(u\cdot v\partial_t\psi-u\cdot vu\cdot\nabla\psi-2u_iv_{i,j}\psi_{,j}+(u_{i,j}v_i+u_iv_{i,j})\psi_{,j}-qu\cdot\nabla\psi\Big)dxdt+$$

$$+\int\limits_{Q_-}v\psi\cdot\nabla pdxdt=$$

$$=\int\limits_{Q_+}\Big(u\cdot v\partial_t\psi-u\cdot vu\cdot\nabla\psi-2u_iv_{i,j}\psi_{,j}-u\cdot v\triangle\psi-(qu+pv)\cdot\nabla\psi\Big)dxdt.$$

As it has been shown in [9] and [7], one may assume that some scaled invariant energy quantities of w are bounded. The same quantities remain to be bounded for $\tilde{u}$ and therefore for u. To be precise, we have

$$A+E+C+D+C_1+D_1+F+H+G=M<\infty, \tag{7.2}$$

where

$$A=\sup_{R>0}\sup_{R^2>t>0}\frac{1}{R}\int\limits_{B(R)}|u(x,t)|^2dx,$$

$$E = \sup_{R>0} \frac{1}{R} \int\limits_{Q_+(R)} |\nabla u(x,t)|^2 dxdt,$$

$$C = \sup_{R>0} \frac{1}{R^2} \int\limits_{Q_+(R)} |u|^3 dxdt, \qquad D = \sup_{R>0} \frac{1}{R^2} \int\limits_{Q_+(R)} |p|^{\frac{3}{2}} dxdt,$$

$$C_1 = \sup_{R>0} \frac{1}{R^{\frac{5}{3}}} \int\limits_{Q_+(R)} |u|^{\frac{10}{3}} dxdt, \qquad D_1 = \sup_{R>0} \frac{1}{R^{\frac{5}{3}}} \int\limits_{Q_+(R)} |p|^{\frac{5}{3}} dxdt,$$

$$F = \sup_{R>0} \frac{1}{R^3} \int\limits_{Q_+(R)} |u|^2 dxdt, \qquad H = \sup_{R>0} \frac{1}{R^{\frac{5}{2}}} \int\limits_{Q_+(R)} |u|^{\frac{5}{2}} dxdt,$$

$$G = \sup_{R>0} \frac{1}{R} \int\limits_{Q_+(R)} |u|^4 dxdt$$

and $Q_+(R) := B(R) \times]0, R^2[$.

We pick $\psi(x,t) = \chi(t)\varphi(x)$. Using simple arguments and smoothness of u and v, we can get rid of χ and have

$$J_R(T) = \int\limits_0^T \int\limits_{\mathbb{R}^3} u \cdot \varphi \operatorname{div} F dxdt = -\int\limits_{\mathbb{R}^3} \varphi(x) u(x,T) \cdot v(x,T) dx +$$

$$+ \int\limits_0^T \int\limits_{\mathbb{R}^3} \Big(u \cdot v u \cdot \nabla\varphi + 2u_i v_{i,j} \varphi_{,j} + u \cdot v \triangle \varphi + (qu + pv) \cdot \nabla \varphi \Big) dxdt.$$

Fix a cut-off function $\varphi(x) = \xi(x/R)$, where $\xi \in C_0^\infty(\mathbb{R}^3)$ with the following properties: $0 \leq \xi \leq 1$, $\xi(x) = 1$ if $|x| \leq 1$, and $\xi(x) = 0$ if $|x| \geq 2$. Our aim is to show that

$$J^1_R(T) = \int\limits_0^T \int\limits_{\mathbb{R}^3} \Big(u \cdot v u \cdot \nabla\varphi + 2u_i v_{i,j} \varphi_{,j} + u \cdot v \triangle \varphi + (qu + pv) \cdot \nabla \varphi \Big) dxdt$$

tends to zero if $R \to \infty$.

Assuming $R^2 > T$, we start with

$$\Big| \int\limits_0^T \int\limits_{\mathbb{R}^3} 2u_i v_{i,j} \varphi_{,j} dxdt \Big| \leq \frac{c}{R} \Big(\int\limits_0^T \int\limits_{B(2R)} |u|^2 dxdt \Big)^{\frac{1}{2}} \Big(\int\limits_0^T \int\limits_{\mathbb{R}^3} |\nabla v|^2 dxdt \Big)^{\frac{1}{2}} \leq$$

$$\leq c\sqrt{A}\sqrt{\frac{T}{R}}\|\nabla v\|_{2,Q_+}\to 0$$

as $R\to\infty$.

Next, we have

$$\Big|\int\limits_0^T\int\limits_{\mathbb{R}^3} u\cdot v\triangle\varphi dxdt\Big|\leq\frac{c}{R^2}\Big(\int\limits_T^0\int\limits_{B(2R)}|u|^2dxdt\Big)^{\frac12}\Big(\int\limits_0^T\int\limits_{\mathbb{R}^3}|v|^2dxdt\Big)^{\frac12}\leq$$

$$\leq c\sqrt{A}\sqrt{\frac{T^2}{R^3}}\|v\|_{2,\infty,Q_+}\to 0$$

as $R\to\infty$.

The third term is estimated as follows:

$$\Big|\int\limits_0^T\int\limits_{\mathbb{R}^3}\Big(u\cdot vu\cdot\nabla\varphi dxdt\Big|\leq$$

$$\leq\frac{c}{R}\Big(\int\limits_0^T\int\limits_{B(2R)}|w|^4dxdt\Big)^{\frac12}\Big(\int\limits_0^T\int\limits_{B(2R)\setminus B(R)}|v|^2dxdt\Big)^{\frac12}\leq$$

$$\leq\frac{c}{\sqrt{R}}\Big(\frac{1}{2R}\int\limits_{Q_+(2R)}|u|^4dxdt\Big)^{\frac12}\Big(\int\limits_0^T\int\limits_{\mathbb{R}^3}|v|^2dxdt\Big)^{\frac12}\leq$$

$$\leq c\sqrt{\frac{GT}{R}}\|v\|_{2,\infty,Q_+}\to 0$$

as $R\to\infty$.

Now, we are going to estimate terms with pressure

$$\Big|\int\limits_0^T\int\limits_{\mathbb{R}^3}pv\cdot\nabla\varphi dxdt\Big|\leq\frac{C}{R}\Big(\int\limits_0^T\int\limits_{B(2R)}|p|^{\frac53}dxdt\Big)^{\frac35}\Big(\int\limits_0^T\int\limits_{B(2R)\setminus B(R)}|v|^{\frac52}dxdt\Big)^{\frac25}\leq$$

$$\leq cD^{\frac35}\Big(\int\limits_0^T\int\limits_{B(2R)\setminus B(R)}|v|^{\frac52}dxdt\Big)^{\frac25}\to 0$$

as $R\to\infty$. The latter is true since the integral

$$\int\limits_0^T\int\limits_{\mathbb{R}^3}|v|^{\frac52}dxdt$$

is finite. Indeed, this follows from the multiplicative inequality

$$\int\limits_0^T\int\limits_{\mathbb{R}^3}|v|^{\frac52}dxdt\leq cT^{\frac58}\|v\|^{\frac74}_{2,\infty,Q_+}\|\nabla v\|^{\frac34}_{2,Q_+}.$$

The most difficult term is the last one. To treat it, we split pressure q into two parts $q=P_1+P_2$ so that

$$\triangle P_1=-\mathrm{div}\,\mathrm{div}\,v\otimes u$$

and

$$\triangle P_2=\mathrm{div}\,\mathrm{div}\,F.$$

As to the second part P_2, we know that it belongs to $L_\infty(0,T;L_2(\mathbb{R}^3))$. This is an immediate consequence of the solution formula

$$P_2(x,t)=\frac13\mathrm{trace}\,F(x,t)-\int\limits_{\mathbb{R}^3}K(x-y):F(y,t)dy,$$

with the kernel $K(x)=\frac{1}{4\pi}\nabla^2\Big(\frac{1}{|x|}\Big)$. Then, we have

$$\Big|\int\limits_0^T\int\limits_{\mathbb{R}^3}P_2u\cdot\nabla\varphi dxdt\Big|\leq\frac{c}{R}\Big(\int\limits_0^T\int\limits_{\mathbb{R}^3}|P_2|^2dxdt\Big)^{\frac12}\Big(\int\limits_0^T\int\limits_{B(2R)}|u|^2dxdt\Big)^{\frac12}\leq$$

$$\leq\sqrt{A}c\sqrt{\frac{T^2}{R}}\|P_2\|_{2,\infty,Q_+}\to0$$

as $R\to\infty$.

Regarding the second part, we are going to use the following decomposition:

$$P_1(x,t)=p_{1R}(x,t)+p_{2R}(x,t)+c_R(t),$$

where

$$p_{1R}(x,t)=-\frac{1}{3}u(x,t)\cdot v(x,t)+\int\limits_{B(3R)}K(x-y):v(y,t)\otimes w(y,t)dy,$$

$$p_{2R}(x,t)=\int\limits_{\mathbb{R}^3\setminus B(3R)}(K(x-y)-K(-y)):v(y,t)\otimes u(y,t)dy,$$

and

$$c_R(t)=\int\limits_{\mathbb{R}^3\setminus B(3R)}K(-y):v(y,t)\otimes w(y,t)dy.$$

First of all, we observe that

$$\int\limits_0^T\int\limits_{\mathbb{R}^3}P_1u\cdot\nabla\varphi dxdt=\int\limits_0^T\int\limits_{\mathbb{R}^3}p_{1R}u\cdot\nabla\varphi dxdt+\int\limits_0^T\int\limits_{\mathbb{R}^3}p_{2R}u\cdot\nabla\varphi dxdt.$$

By the theory of singular integrals,

$$\int\limits_{B(3R)}|p_{1R}|^{\frac{4}{3}}dx\leq c\int\limits_{B(3R)}|u|^{\frac{4}{3}}|v|^{\frac{4}{3}}dx$$

and thus

$$\int\limits_0^T\int\limits_{B(3R)}|p_{1R}|^{\frac{4}{3}}dxdt\leq c\Big(\int\limits_0^T\int\limits_{B(3R)}|u|^4dxdt\Big)^{\frac{1}{3}}\Big(\int\limits_0^T\int\limits_{B(3R)}|v|^2dxdt\Big)^{\frac{2}{3}}\leq$$

$$\leq cR^{\frac{1}{3}}G^{\frac{1}{3}}T^{\frac{2}{3}}\|v\|^{\frac{4}{3}}_{2,\infty,Q_+}.$$

So,

$$\Big|\int\limits_0^T\int\limits_{\mathbb{R}^3}p_{1R}u\cdot\nabla\varphi dxdt\Big|\leq\frac{c}{R}\Big(\int\limits_T^0\int\limits_{B(2R)}|p_{1R}|^{\frac{4}{3}}dxdt\Big)^{\frac{3}{4}}\Big(\int\limits_T^0\int\limits_{B(3R)}|u|^4dxdt\Big)^{\frac{1}{4}}\leq$$

$$\leq\frac{c}{R}R^{\frac{1}{4}}G^{\frac{1}{4}}T^{\frac{1}{2}}\|v\|_{2,\infty,Q_+}R^{\frac{1}{4}}G^{\frac{1}{4}}\to 0$$

as $R\to\infty$.

Assuming that $R < |x| < 2R$ and $0 < t < T$, we have for the second counterpart the following estimate

$$|p_{2R}(x,t)| \leq c \int\limits_{\mathbb{R}^3 \setminus B(3R)} \frac{|x|}{|y|^4} |u(y,t)||v(y,t)|dy \leq$$

$$\leq cR \sum_{k=0}^{\infty} \frac{1}{(R2^k)^4} \int\limits_{R2^k < |y| < R2^{k+1}} |u(y,t)||v(y,t)|dy \leq$$

$$\leq cR \sum_{k=0}^{\infty} \frac{1}{(R2^k)^4} \Big(\int\limits_{B(R2^{k+1})} |u(y,t)|^2 dy \Big)^{\frac{1}{2}} \Big(\int\limits_{B(R2^{k+1})} |v(y,t)|^2 dy \Big)^{\frac{1}{2}} \leq$$

$$\leq cR \Big(\int\limits_{\mathbb{R}^3} |v(y,t)|^2 dy \Big)^{\frac{1}{2}} \sum_{k=0}^{\infty} \frac{1}{(R2^k)^4} (R2^{k+1})^{\frac{1}{2}} A^{\frac{1}{2}} \leq$$

$$\leq \sqrt{A} \frac{c}{R^{\frac{5}{2}}} \|v\|_{2,\infty,Q_+}.$$

Then,

$$\Big| \int\limits_0^T \int\limits_{\mathbb{R}^3} p_{2R} w \cdot \nabla \varphi dx dt \Big| \leq \frac{c}{R} \int\limits_0^T \sqrt{A} \frac{1}{R^{\frac{5}{2}}} \|v\|_{2,\infty,Q_+} \int\limits_{B(2R)} |u(x,t)| dx dt \leq$$

$$\leq \sqrt{A} \frac{c}{R^{\frac{7}{2}}} |B(2R)|^{\frac{1}{2}} \|v\|_{2,\infty,Q_+} \int\limits_0^T \Big(\int\limits_{B(2R)} |u(y,t)|^2 dy \Big)^{\frac{1}{2}} dt$$

$$\leq (-AT) \frac{c}{R^{\frac{3}{2}}} \|v\|_{2,\infty,Q_+} \to 0$$

as $R \to \infty$. So, finally, we have

$$\int\limits_0^T \int\limits_{\mathbb{R}^3} u \cdot \operatorname{div} F dx dt = - \lim_{R \to \infty} \int\limits_{\mathbb{R}^3} \varphi(x) u(x,T) \cdot v(x,T) dx.$$

Taking into account $u(\cdot,T) \cdot v(\cdot,T) \in L_1(\mathbb{R}^3)$, see (1.5), we conclude that

$$\int\limits_0^T \int\limits_{\mathbb{R}^3} u \cdot \operatorname{div} F dx dt = - \int\limits_{\mathbb{R}^3} u(x,T) \cdot v(x,T) dx.$$

for any $T > 0$.

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