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Pricing Spread Options using Matched Asymptotic Expansions

Dissertation

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Abstract

This document deals with approximating spread options prices using Matched Asymptotic Expansions techniques on the correlation. More precisely, it deals with spreads options on assets that are highly correlated ($\rho \sim 1$), which is most commonly observed in Oil Markets (Crude Oil vs. Gasoline for example). We will first start by applying this methodology to exchange options before generalizing our results to spread options. Then we are going to describe an alternative approach of pricing spread options by approximating the bivariate normal distribution. Finally, we will see how we can apply our methodology to the case where we have more than two assets.

Contents

1 Introduction

Spread options are contracts that are written on the difference between two[1] underlying assets S_1 and S_2. The buyer has the right at maturity to buy/sell the quantity $(S_1 - S_2)$ at a given strike K. These types of options have been actively traded this past decade in different classes of assets and more particularly in Fixed Income and Commodities Markets. For example, in the Interest Rates Market, one of the most popular products are *CMS (Constant Maturity Swap) spreads options*. These are contracts written on the difference between two Swap rates of constant maturities and they allow the buyer to hedge against/speculate on the steepening or the flattening of the yield curve. Several spread options are also traded in the Agricultural Futures markets. As an example, *crush spread options* which are traded on the Chicago Board of Trade (CBOT). The underlying indexes are future contracts of soybean and its derivatives which are soybean oil and soybean meal. However, spread options are predominant in Energy Markets. In the Electricity Market, the most traded contracts are *spark spreads* which represent the theoretical margin of a power plant. Indeed, it is a spread between the electricity sold by a generator and the cost of the fuel (or natural gaz, coal...) used to generate it. The most popular spark spreads have payoff at maturity T in the form

$$(F_e(T) - K_e F_g(T) - K)^+ \tag{1.1}$$

where F_e and F_g are respectively the prices of future contracts on electricity and natural gas and K_e is the efficiency factor of a power plant. F_e and F_g have different units. The first is expressed in \$/MWh whereas the second is expressed in \$/Btu. The efficiency factor is expressed in Btu/MWh, therefore the spread option will be given in \$/MWh.

However, in this paper we will focus our interest on Oil Markets and more specifically one of the most frequently quoted spread options which are *crack spreads*. A crack spread represents the differential between the price of crude oil and petroleum products (gasoline or heating oil). These spread options were introduced by the NYMEX (New York Mercantile Exchange) in 1994 and are quoted in \$/barrel. A typical payoff is given of the form of

$$(S_{Pe}(T) - S_{CO}(T) - K)^+ \tag{1.2}$$

where S_{Pe} and S_{CO} represent respectively the prices of Petroleum products and Crude Oil. However, there are different types of crack spreads. For example, the 3:2:1 crack spread that include three contracts of crude oil, two contracts of unleaded gasoline and one contract of heating oil. The payoff in that case in given in the following form

$$\left(\frac{2}{3} S_{UG}(T) + \frac{1}{3} S_{HO}(T) - S_{CO}(T) - K\right)^+ \tag{1.3}$$

where S_{UG}, S_{HO} and S_{CO} respectively represent the prices of unleaded gasoline, heating oil and crude oil. There are also contracts that involve crude oil and one or the other of heating oil and unleaded gasoline. In our study, we will concentrate more on the latter

[1]There are many spread options that take as underlying more than two assets. We choose two assets as a simplified framework.

which represents a spread between two underlying assets.

Pricing spread options has been a challenge among academics and practitioners for many years. Indeed, an exact closed form solution is not available. However, many approximations have been proposed throughout the years. The first attempt was introduced by Bachelier where he modelled the spread using a Arithmetic Brownian motion. Many other approximations have then followed. One can note for example Kirk (1995), Carmona and Durrleman (2003), Li, Deng and Zhou (2007). Nevertheless, in the case where the strike $K = 0$ (exchange option), we have an exact closed form solution that has been introduced by Margrabe (1978).

In our study, we are going to place ourselves in a different framework from the above mentioned approximations which is pricing using Matched Asymptotic Expansions (also known as perturbation techniques). In fact, we want to model the price of a crack spread option by assuming that the the crude oil and the gasoline are very correlated. In that case, we can apply a perturbation around the value of correlation $\rho = 1$. But before doing this, we first need to check if our assumption is correct. Let us look at the historical correlation chart between gasoline and crude oil. The following chart[2] represents the historical correlation between 01/2004 and 07/2007

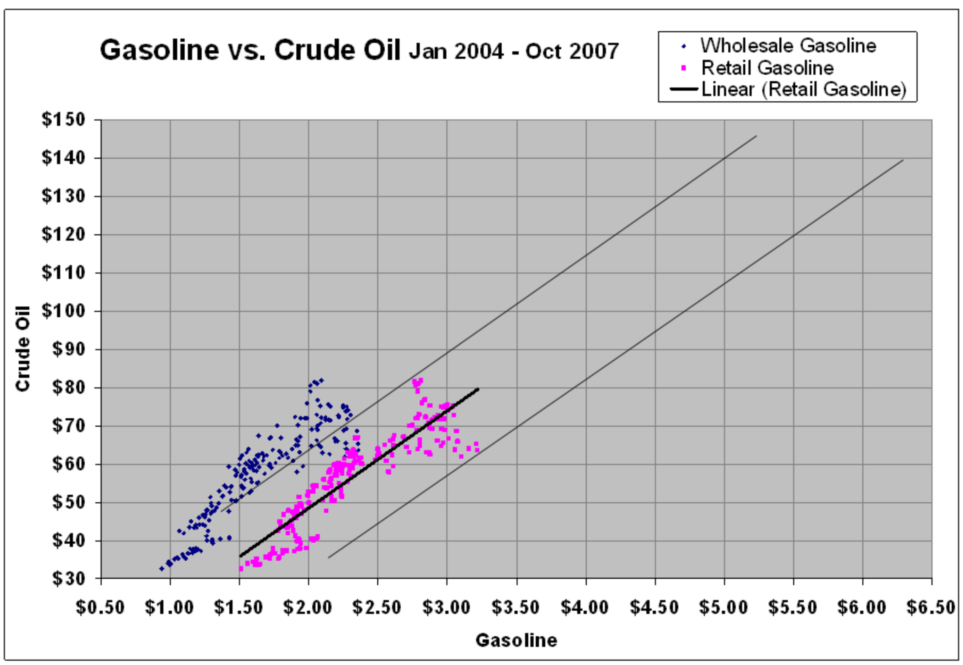

Figure 1.1. Crude Oil vs. Gasoline

We can see that, as expected, crude oil and gasoline are very correlated. In our framework, we are going to try to find approximations for crack spreads when the correlation is in the interval $[0.9, 1]$ which will be the domain of validity of our formula. We will first

[2]Chart provided by Crude Oil Global Oil Production http://crudeoil.files.wordpress.com/

start by applying an asymptotic expansion on the correlation in the two dimensional Black & Scholes PDE for spread options. Then we are going to study separately the cases $K = 0$ and $K \neq 0$. Further more, we are going to introduce an alternative method by approximating the bivariate normal distribution. Finally, we will see how we can generalize our results to the case where we have two or more assets.

2 Mathematical Setup

We consider that we have a market with two assets S_1 and S_2 evolving in time. We suppose that the two assets follow a geometric Brownian motion, i.e:

$$\frac{dS_i(t)}{S_i(t)} = \mu_i dt + \sigma_i dW_i(t), \qquad i = 1, 2. \tag{2.4}$$

where W_1 and W_2 are two correlated Brownian Motions with correlation ρ. Using Ito's formula, we can get explicit expressions for S_1 and S_2:

$$S_i(t) = S_i(0) \exp\left[(\mu_i - \frac{1}{2}\sigma_i^2)t + \sigma_i W_i(t)\right], \qquad i = 1, 2. \tag{2.5}$$

Our aim is to price a Spread Option, i.e an option which has the following payoff at maturity T (a call in this case):

$$(S_1(T) - S_2(T) - K)^+ = (S_1(T) - S_2(T) - K)\,\mathbf{1}_{\{S_1(T) - S_2(T) > K\}} \tag{2.6}$$

where K is the strike. The first intuition is to compute the expectation of the payoff under the risk-neutral $\mathbb{Q}$ measure at time t:

$$V(t, S_1, S_2) = e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}\left[(S_1(T) - S_2(T) - K)^+ \,|\mathcal{F}_t\right] = e^{-r(T-t)}\int\int (s_2 - s_1 - K)^+ h_T(s_1, s_2) ds_1 ds_2 \tag{2.7}$$

where h_T is the joint density function of $S_1(T)$ and $S_2(T)$. However, this formula involves using a numerical integration of the inner integral which is not very practical. Thus we can see that it is not straight forward to obtain an exact closed form solution. In what follows, we will use a PDE approach and we will apply an asymptotic expansion on the correlation in order to try to find an approximation for the price of Spread Options.

3 Pricing Spread Options using Matched Asymptotic Expansions

Under the standard Black and Scholes replication arguments, the price V of a Spread Option satisfies the following PDE:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\left(\sigma_1^2 S_1^2 \frac{\partial^2 V}{\partial S_1^2} + 2\rho\sigma_1\sigma_2 S_1 S_2 \frac{\partial^2 V}{\partial S_1 \partial S_2} + \sigma_2^2 S_2^2 \frac{\partial^2 V}{\partial S_2^2}\right) + r\left(S_1 \frac{\partial V}{\partial S_1} + S_2 \frac{\partial V}{\partial S_2}\right) - rV = 0$$

$$V(T, S_1, S_2) = (S_1(T) - S_2(T) - K)^+ \tag{3.8}$$

The *Feynman-Kac* representation tells us that solving this PDE gives the same result as computing the expectation of the payoff.

We would like now to solve this PDE asymptotically using an expansion on the correlation term. We place ourselves in a framework where the assets S_1 and S_2 are very correlated i.e $\rho \sim 1$. If we now let ϵ such that $\epsilon^2 \ll 1$, we can write $\rho = 1 - \epsilon^2$. The new PDE will then be:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\left(\sigma_1^2 S_1^2 \frac{\partial^2 V}{\partial S_1^2} + 2\sigma_1\sigma_2 S_1 S_2 \frac{\partial^2 V}{\partial S_1 \partial S_2} + \sigma_2^2 S_2^2 \frac{\partial^2 V}{\partial S_2^2}\right) + r\left(S_1 \frac{\partial V}{\partial S_1} + S_2 \frac{\partial V}{\partial S_2}\right) - rV = \epsilon^2 \sigma_1\sigma_2 S_1 S_2 \frac{\partial^2 V}{\partial S_1 \partial S_2}$$

$$V(T, S_1, S_2) = (S_1(T) - S_2(T) - K)^+ \tag{3.9}$$

We will then have a regular expansion:

$$V(t, S_1, S_2) \sim V_0(t, S_1, S_2) + \epsilon^2 V_1(t, S_1, S_2) + ... \tag{3.10}$$

where V_0 and V_1 satisfy the following:

$$\frac{\partial V_0}{\partial t} + \frac{1}{2}\left(\sigma_1^2 S_1^2 \frac{\partial^2 V_0}{\partial S_1^2} + 2\sigma_1\sigma_2 S_1 S_2 \frac{\partial^2 V_0}{\partial S_1 \partial S_2} + \sigma_2^2 S_2^2 \frac{\partial^2 V_0}{\partial S_2^2}\right) + r\left(S_1 \frac{\partial V_0}{\partial S_1} + S_2 \frac{\partial V_0}{\partial S_2}\right) - rV_0 = 0$$

$$V_0(T, S_1, S_2) = (S_1(T) - S_2(T) - K)^+ \tag{3.11}$$

and

$$\frac{\partial V_1}{\partial t} + \frac{1}{2}\left(\sigma_1^2 S_1^2 \frac{\partial^2 V_1}{\partial S_1^2} + 2\sigma_1\sigma_2 S_1 S_2 \frac{\partial^2 V_1}{\partial S_1 \partial S_2} + \sigma_2^2 S_2^2 \frac{\partial^2 V_1}{\partial S_2^2}\right) + r\left(S_1 \frac{\partial V_1}{\partial S_1} + S_2 \frac{\partial V_1}{\partial S_2}\right) - rV_1 = \sigma_1\sigma_2 S_1 S_2 \frac{\partial^2 V_0}{\partial S_1 \partial S_2}$$

$$V_1(T, S_1, S_2) = 0 \tag{3.12}$$

One can see that the above equations are not easily solvable unless we introduce a change of variable, which will allow us to eliminate the cross derivatives terms.

3.1 Transformation of the 2-D Black Scholes PDE

In this section, we want to find a similarity solution of the PDE (5.55) in order to eliminate the cross derivatives terms in S_1 and S_2.

Proposition 1 *Let $V(t, S_1, S_2) = F(t, u, v)$, where*

$$u = S_1^{\frac{\Sigma}{\sigma_1}} S_2^{\frac{\Sigma}{\sigma_1}} e^{\left(r(-1+\frac{\Sigma}{\sigma_1}+\frac{\Sigma}{\sigma_2})+\frac{\Sigma}{2}(4\Sigma-\sigma_1-\sigma_2)\right)(T-t)}, \quad v = S_1^{\frac{\Sigma}{\sigma_1}} S_2^{\frac{-\Sigma}{\sigma_1}} e^{\left(r(\frac{\Sigma}{\sigma_1}-\frac{\Sigma}{\sigma_2})+\frac{\Sigma}{2}(\sigma_2-\sigma_1)\right)(T-t)}$$

and Σ a constant which has a volatility dimension[3]*. Then the PDE satisfied by F is :*

$$\frac{\partial F}{\partial t} + (1+\rho)\Sigma^2 u^2 \frac{\partial^2 F}{\partial u^2} + (1-\rho)\Sigma^2 v^2 \frac{\partial^2 F}{\partial v^2} + (r-(1-\rho)\Sigma^2)u\frac{\partial F}{\partial u} + (1-\rho)\Sigma^2 v \frac{\partial F}{\partial v} - rF = 0$$

$$F(T,u,v) = \left(u^{\frac{\sigma_1}{2\Sigma}} v^{\frac{\sigma_1}{2\Sigma}} - u^{\frac{\sigma_2}{2\Sigma}} v^{-\frac{\sigma_2}{2\Sigma}} - K\right)^+ \tag{3.13}$$

Remark 1 *Note that $1+\rho$ and $1-\rho$ are the eigenvalues of the correlation matrix*

$$\mathcal{C} = \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix} \tag{3.14}$$

<u>*Proof*</u>:
Our goal is to diagonalize the 2D-BSPDE. Thus if we let $V(t,S_1,S_2) = F(t,u,v)$, we want V and F to satisfy the following system of equations for some Σ:

$$\begin{cases} \sigma_1 S_1 \dfrac{\partial V}{\partial S_1} = \Sigma u \dfrac{\partial F}{\partial u} + \Sigma v \dfrac{\partial F}{\partial v} \\ \sigma_2 S_2 \dfrac{\partial V}{\partial S_2} = \Sigma u \dfrac{\partial F}{\partial u} - \Sigma v \dfrac{\partial F}{\partial v} \end{cases}$$

A solution to this system is $u = C_u(t) S_1^{\frac{\Sigma}{\sigma_1}} S_2^{\frac{\Sigma}{\sigma_1}}$ and $v = C_v(t) S_1^{\frac{\Sigma}{\sigma_2}} S_2^{-\frac{\Sigma}{\sigma_{12}}}$ where C_u and C_v are some functions of time to be determined. Let us first set $C_u = C_v = 1$. Then in this case we have:

$$\begin{aligned}
\sigma_1^2 S_1^2 \frac{\partial^2 V}{\partial S_1^2} &= \sigma_1^2 S_1^2 \frac{\partial}{\partial S_1}\left(\frac{\partial V}{\partial S_1}\right) \\
&= \sigma_1^2 S_1^2 \frac{\partial}{\partial S_1}\left(\frac{\Sigma u}{\sigma_1 S_1}\frac{\partial F}{\partial u} + \frac{\Sigma v}{\sigma_1 S_1}\frac{\partial F}{\partial v}\right) \\
&= \sigma_1^2 S_1^2 \left(\left(\frac{\Sigma}{\sigma_1} - 1\right)\frac{\Sigma u}{\sigma_1 S_1^2}\frac{\partial F}{\partial u} + \frac{\Sigma u}{\sigma_1 S_1}\frac{\partial}{\partial S_1}\left(\frac{\partial F}{\partial u}\right) + \left(\frac{\Sigma}{\sigma_1} - 1\right)\frac{\Sigma v}{\sigma_1 S_1^2}\frac{\partial F}{\partial v}\right. \\
&\qquad \left. + \frac{\Sigma v}{\sigma_1 S_1}\frac{\partial}{\partial S_1}\left(\frac{\partial F}{\partial v}\right)\right) \\
&= \Sigma\left(\Sigma - \sigma_1\right) u \frac{\partial F}{\partial u} + \Sigma\left(\Sigma - \sigma_1\right) v \frac{\partial F}{\partial v} + \Sigma \sigma_1 S_1 u \left(\frac{\partial^2 F}{\partial u^2}\frac{\partial u}{\partial S_1} + \frac{\partial^2 F}{\partial u \partial v}\frac{\partial v}{\partial S_1}\right) \\
&\qquad + \Sigma \sigma_1 S_1 v \left(\frac{\partial^2 F}{\partial v^2}\frac{\partial v}{\partial S_1} + \frac{\partial^2 F}{\partial u \partial v}\frac{\partial u}{\partial S_1}\right) \\
&= \Sigma\left(\Sigma - \sigma_1\right) u \frac{\partial F}{\partial u} + \Sigma\left(\Sigma - \sigma_1\right) v \frac{\partial F}{\partial v} + \Sigma^2 u^2 \frac{\partial^2 F}{\partial u^2} + \Sigma^2 v^2 \frac{\partial^2 F}{\partial v^2} + 2\Sigma^2 uv \frac{\partial^2 F}{\partial u \partial v}
\end{aligned}$$

[3] The role of Σ is to make sure that we have a dimensionally consistent PDE. We can assign any value to it.

$$\begin{aligned}
\sigma_2^2 S_2^2 \frac{\partial^2 V}{\partial S_2^2} &= \sigma_2^2 S_2^2 \frac{\partial}{\partial S_2}\left(\frac{\partial V}{\partial S_2}\right)\\
&= \sigma_2^2 S_2^2 \frac{\partial}{\partial S_2}\left(\frac{\Sigma u}{\sigma_2 S_2}\frac{\partial F}{\partial u} - \frac{\Sigma v}{\sigma_2 S_2}\frac{\partial F}{\partial v}\right)\\
&= \sigma_2^2 S_2^2 \left(\left(\frac{\Sigma}{\sigma_2}-1\right)\frac{\Sigma u}{\sigma_2 S_2^2}\frac{\partial F}{\partial u} + \frac{\Sigma u}{\sigma_2 S_2}\frac{\partial}{\partial S_2}\left(\frac{\partial F}{\partial u}\right) + \left(\frac{\Sigma}{\sigma_2}+1\right)\frac{\Sigma v}{\sigma_2 S_2^2}\frac{\partial F}{\partial v}\right.\\
&\qquad \left. - \frac{\Sigma v}{\sigma_2 S_2}\frac{\partial}{\partial S_2}\left(\frac{\partial F}{\partial v}\right)\right)\\
&= \Sigma\left(\Sigma-\sigma_2\right)u\frac{\partial F}{\partial u} + \Sigma\left(\Sigma+\sigma_2\right)v\frac{\partial F}{\partial v} + \Sigma\sigma_2 S_2 u\left(\frac{\partial^2 F}{\partial u^2}\frac{\partial u}{\partial S_2} + \frac{\partial^2 F}{\partial u\partial v}\frac{\partial v}{\partial S_2}\right)\\
&\qquad - \Sigma\sigma_2 S_2 v\left(\frac{\partial^2 F}{\partial v^2}\frac{\partial v}{\partial S_2} + \frac{\partial^2 F}{\partial u\partial v}\frac{\partial u}{\partial S_2}\right)\\
&= \Sigma\left(\Sigma-\sigma_2\right)u\frac{\partial F}{\partial u} + \Sigma\left(\Sigma+\sigma_2\right)v\frac{\partial F}{\partial v} + \Sigma^2 u^2\frac{\partial^2 F}{\partial u^2} + \Sigma^2 v^2\frac{\partial^2 F}{\partial v^2} - 2\Sigma^2 uv\frac{\partial^2 F}{\partial u\partial v}
\end{aligned}$$

$$\begin{aligned}
\sigma_1\sigma_2 S_1 S_2 \frac{\partial^2 V}{\partial S_1 S_2} &= \sigma_1\sigma_2 S_1 S_2 \frac{\partial}{\partial S_2}\left(\frac{\partial V}{\partial S_1}\right)\\
&= \sigma_1\sigma_2 S_1 S_2 \frac{\partial}{\partial S_2}\left(\frac{\Sigma u}{\sigma_1 S_1}\frac{\partial F}{\partial u} + \frac{\Sigma v}{\sigma_1 S_1}\frac{\partial F}{\partial v}\right)\\
&= \sigma_1\sigma_2 S_1 S_2 \left(\frac{\Sigma^2 u}{\sigma_1\sigma_2 S_1 S_2}\frac{\partial F}{\partial u} + \frac{\Sigma u}{\sigma_1 S_1}\frac{\partial}{\partial S_2}\left(\frac{\partial F}{\partial u}\right) - \frac{\Sigma^2 v}{\sigma_1\sigma_2 S_1 S_2}\frac{\partial F}{\partial v}\right.\\
&\qquad \left. + \frac{\Sigma v}{\sigma_1 S_1}\frac{\partial}{\partial S_2}\left(\frac{\partial F}{\partial v}\right)\right)\\
&= \Sigma^2 u\frac{\partial F}{\partial u} - \Sigma^2 v\frac{\partial F}{\partial v} + \Sigma\sigma_2 S_2 u\left(\frac{\partial^2 F}{\partial u^2}\frac{\partial u}{\partial S_2} + \frac{\partial^2 F}{\partial u\partial v}\frac{\partial v}{\partial S_2}\right)\\
&\qquad + \Sigma\sigma_2 S_2 v\left(\frac{\partial^2 F}{\partial v^2}\frac{\partial v}{\partial S_2} + \frac{\partial^2 F}{\partial u\partial v}\frac{\partial u}{\partial S_2}\right)\\
&= \Sigma^2 u\frac{\partial F}{\partial u} - \Sigma^2 v\frac{\partial F}{\partial v} + \Sigma^2 u^2\frac{\partial^2 F}{\partial u^2} - \Sigma^2 v^2\frac{\partial^2 F}{\partial v^2}
\end{aligned}$$

Thus the BSPDE (5.55) under the above change of variable becomes:

$$\begin{aligned}
\frac{\partial F}{\partial t} &+ (1+\rho)\Sigma^2 u^2\frac{\partial^2 F}{\partial u^2} + (1-\rho)\Sigma^2 v^2\frac{\partial^2 F}{\partial v^2} + \left(r\left(\frac{\Sigma}{\sigma_1}+\frac{\Sigma}{\sigma_2}\right) + \frac{\Sigma}{2}(2\Sigma-\sigma_1-\sigma_2)\Sigma^2 + \rho\Sigma^2\right)u\frac{\partial F}{\partial u}\\
&+ \left(r\left(\frac{\Sigma}{\sigma_1}-\frac{\Sigma}{\sigma_2}\right) + \frac{\Sigma}{2}(2\Sigma+\sigma_2-\sigma_1)\Sigma^2 - \rho\Sigma^2\right)v\frac{\partial F}{\partial v} - rF = 0
\end{aligned}$$

If we now let

$$\begin{cases}
C_u(t) = \exp\left[\left(r(-1+\frac{\Sigma}{\sigma_1}+\frac{\Sigma}{\sigma_2}) + \frac{\Sigma}{2}(4\Sigma-\sigma_1-\sigma_2)\right)(T-t)\right]\\
C_v(t) = \exp\left[\left(r(\frac{\Sigma}{\sigma_1}-\frac{\Sigma}{\sigma_2}) + \frac{\Sigma}{2}(\sigma_2-\sigma_1)\right)(T-t)\right]
\end{cases}$$

then we get the following equation:

$$\frac{\partial F}{\partial t}+(1+\rho)\Sigma^2u^2\frac{\partial^2 F}{\partial u^2}+(1-\rho)\Sigma^2v^2\frac{\partial^2 F}{\partial v^2}+(r-(1-\rho)\Sigma^2)u\frac{\partial F}{\partial u}+(1-\rho)\Sigma^2v\frac{\partial F}{\partial v}-rF=0$$

Also, at time $t=T$ we have $C_v(T)=C_v(T)=1$, hence $S_1(T)=u^{\frac{\sigma_1}{2\Sigma}}v^{\frac{\sigma_1}{2\Sigma}}$ and $S_2(T)=u^{\frac{\sigma_2}{2\Sigma}}v^{-\frac{\sigma_2}{2\Sigma}}$, consequently the terminal condition of the above PDE is:

$$F(T,u,v)=\left(u^{\frac{\sigma_1}{2\Sigma}}v^{\frac{\sigma_1}{2\Sigma}}-u^{\frac{\sigma_2}{2\Sigma}}v^{-\frac{\sigma_2}{2\Sigma}}-K\right)^+$$

Which is the result we have stated earlier.

Now that we have found this new PDE, let us again set $\rho=1-\epsilon^2$, in this case we get

$$\frac{\partial F}{\partial t}+2\Sigma^2u^2\frac{\partial^2 F}{\partial u^2}+ru\frac{\partial F}{\partial u}-rF=\epsilon^2\Sigma^2\left(u^2\frac{\partial^2 F}{\partial u^2}-v^2\frac{\partial^2 F}{\partial v^2}+u\frac{\partial F}{\partial u}-v\frac{\partial F}{\partial v}\right) \tag{3.15}$$

If we consider the following expansion

$$F(t,u,v)\sim F_0(t,u,v)+\epsilon^2F_1(t,u,v)+... \tag{3.16}$$

then we get for F_0 and F_1

$$\begin{aligned}&\frac{\partial F_0}{\partial t}+2\Sigma^2u^2\frac{\partial^2 F_0}{\partial u^2}+ru\frac{\partial F_0}{\partial u}-rF_0=0\\&F_0(T,u,v)=\left(u^{\frac{\sigma_1}{2\Sigma}}v^{\frac{\sigma_1}{2\Sigma}}-u^{\frac{\sigma_2}{2\Sigma}}v^{-\frac{\sigma_2}{2\Sigma}}-K\right)^+\end{aligned} \tag{3.17}$$

and

$$\begin{aligned}&\frac{\partial F_1}{\partial t}+2\Sigma^2u^2\frac{\partial^2 F_1}{\partial u^2}+ru\frac{\partial F_1}{\partial u}-rF_1=\Sigma^2\left(u^2\frac{\partial^2 F_0}{\partial u^2}-v^2\frac{\partial^2 F_0}{\partial v^2}+u\frac{\partial F_0}{\partial u}-v\frac{\partial F_0}{\partial v}\right)\\&F_1(T,u,v)=0\end{aligned} \tag{3.18}$$

We notice that the equation of F_0 is a one dimensional Black Scholes PDE with u as the underlying (with a volatility of 2Σ). The PDE does not depend on v, thus v can be treated as a constant equal to its initial value. Once we have solved the equation of F_0, we can calculate its first and second derivatives with respect to u and v and use *Feynman-Kac generalized formula* to get the value of F_1. We will first start by looking at the case where $K=0$ which will allow us to get an exact closed formula before studying the general case i.e when $K\neq 0$.

3.2 Pricing formula in the case $K=0$

Proposition 2 *In the case $K=0$, we have the following closed formula for F_0 (when $\sigma_1\neq\sigma_2$)*

$$F_0(t,u,v)=v_t^{\frac{\sigma_1}{2\Sigma}}u_t^{\frac{\sigma_1}{2\Sigma}}e^{\left(\left(\frac{\sigma_1}{2\Sigma}-1\right)r+\sigma_1\left(\frac{\sigma_1}{2}-\Sigma\right)\right)(T-t)}N(\delta d_1)-v_t^{-\frac{\sigma_2}{2\Sigma}}u_t^{\frac{\sigma_2}{2\Sigma}}e^{\left(\left(\frac{\sigma_2}{2\Sigma}-1\right)r+\sigma_2\left(\frac{\sigma_2}{2}-\Sigma\right)\right)(T-t)}N(\delta d_2) \tag{3.19}$$

where

$$d_i = \frac{\log(u_t) + \frac{\sigma_1+\sigma_2}{\sigma_1-\sigma_2}\log(v_t) + \left(r + 2\left(\frac{\sigma_i}{\Sigma} - 1\right)\Sigma^2\right)(T-t)}{2\Sigma\sqrt{T-t}}, \qquad i = 1, 2.$$

and

$$\delta = \begin{cases} 1 & \text{if } \sigma_1 > \sigma_2 \\ -1 & \text{if } \sigma_1 < \sigma_2 \end{cases}$$

and

$$N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{s^2}{2}} ds$$

Proof:

From equation (3.17), we know that the payoff of F_0 is given by:

$$F_0(T,u,v) = \left(u_T^{\frac{\sigma_1}{2\Sigma}} v_t^{\frac{\sigma_1}{2\Sigma}} - u_T^{\frac{\sigma_2}{2\Sigma}} v_t^{-\frac{\sigma_2}{2\Sigma}}\right)^+ = \left(u_T^{\frac{\sigma_1}{2\Sigma}} v_t^{\frac{\sigma_1}{2\Sigma}} - u_T^{\frac{\sigma_2}{2\Sigma}} v_t^{-\frac{\sigma_2}{2\Sigma}}\right) \mathbf{1}_{\left\{u_T^{\frac{\sigma_1}{2\Sigma}} v_t^{\frac{\sigma_1}{2\Sigma}} > u_T^{\frac{\sigma_2}{2\Sigma}} v_t^{-\frac{\sigma_2}{2\Sigma}}\right\}}$$

and the exercise region can be written

$$\left\{u_T^{\frac{\sigma_1}{2\Sigma}} v_t^{\frac{\sigma_1}{2\Sigma}} > u_T^{\frac{\sigma_2}{2\Sigma}} v_t^{-\frac{\sigma_2}{2\Sigma}}\right\} = \left\{u_T^{\frac{\sigma_1-\sigma_2}{2\Sigma}} > v_t^{\frac{-\sigma_1-\sigma_2}{2\Sigma}}\right\} = \begin{cases} \left\{u_T > v_t^{-\frac{\sigma_1+\sigma_2}{\sigma_1-\sigma_2}}\right\} \text{if } \sigma_1 > \sigma_2 \\ \left\{u_T < v_t^{-\frac{\sigma_1+\sigma_2}{\sigma_1-\sigma_2}}\right\} \text{if } \sigma_1 < \sigma_2 \end{cases}$$

Note that if we solve the problem for the case $\sigma_1 > \sigma_2$, we can find the solution for the other case using the fact that

$$\mathbf{1}_{\left\{u_T < v_t^{-\frac{\sigma_1+\sigma_2}{\sigma_1-\sigma_2}}\right\}} = 1 - \mathbf{1}_{\left\{u_T > v_t^{-\frac{\sigma_1+\sigma_2}{\sigma_1-\sigma_2}}\right\}};$$

thus we can restrict our proof to the case where $\sigma_1 > \sigma_2$. In this case if we let $X = v_t^{-\frac{\sigma_1+\sigma_2}{\sigma_1-\sigma_2}}$, then the payoff of F_0 is:

$$F_0(T,u,v) = v_t^{\frac{\sigma_1}{2\Sigma}} u_T^{\frac{\sigma_1}{2\Sigma}} \mathbf{1}_{\{u_T > X\}} - v_t^{-\frac{\sigma_2}{2\Sigma}} u_T^{\frac{\sigma_2}{2\Sigma}} \mathbf{1}_{\{u_T > X\}}$$

Also, from equation (3.17), we know that u_t follows a geometric Brownian motion under the risk neutral measure $\mathbb{Q}$ with volatility 2Σ, i.e

$$\frac{du_t}{u_t} = rdt + 2\Sigma dW_t^{\mathbb{Q}}$$

At this stage, we will introduce the following Lemma:

Lemma 1 *Let $(S_t)_t$ be a process following a geometric Brownian motion,*

$$\frac{dS_t}{S_t} = rdt + \sigma dW_t^{\mathbb{Q}}$$

Then the value V of an option whose payoff at time T is $V(T,S) = S_T^a \mathbf{1}_{\{S_T>K\}}$ (where a is constant) is given by (at time t)

$$V(t,S) = S_t^a e^{\left((a-1)r+\frac{a}{2}(a-1)\sigma^2\right)(T-t)} N(d) \tag{3.20}$$

where

$$d = \frac{\log\left(\frac{S}{K}\right) + \left(r + (2a-1)\frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}$$

<u>*Proof of lemma 1:*</u>

Using Ito's lemma, we know that

$$S_T = S_t \exp\left[(r - \frac{1}{2}\sigma^2)t + \sigma W_t^{\mathbb{Q}}\right]$$

Thus, by computing the expectation of the payoff under the risk neutral measure $\mathbb{Q}$, we get

$$\begin{aligned}
V(t,S) &= e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}\left[S_T^a \mathbf{1}_{\{S_T>K\}} | \mathcal{F}_t\right] \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\frac{\log(S/K)+(r-\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}}^{\infty} S_t^a e^{\left(-r+a(r-\frac{1}{2}\sigma^2)\right)(T-t)+a\sigma\sqrt{T-t}x} e^{-\frac{x^2}{2}} dx \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\frac{\log(S/K)+(r-\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}-a\sigma\sqrt{T-t}}^{\infty} S_t^a e^{\left(-r+a(r-\frac{1}{2}\sigma^2+\frac{1}{2}a^2\sigma^2)\right)(T-t)} e^{-\frac{y^2}{2}} dy \\
&= S_t^a e^{\left((a-1)r+\frac{a}{2}(a-1)\sigma^2\right)(T-t)} N\left(\frac{\log\left(\frac{S}{K}\right) + \left(r - \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}} + a\sigma\sqrt{T-t}\right) \\
&= S_t^a e^{\left((a-1)r+\frac{a}{2}(a-1)\sigma^2\right)(T-t)} N\left(\frac{\log\left(\frac{S}{K}\right) + \left(r + (2a-1)\frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}\right)
\end{aligned}$$

By applying this formula to F_0 (by successively taking $a = \dfrac{\sigma_1}{2\Sigma}$ and $a = \dfrac{\sigma_2}{2\Sigma}$), we get the result stated in the Proposition.

Remark 2 *Note that if we replace u and v by their respective values, we get exactly Margrabe's formula with $\rho = 1$, i.e*

$$F_0(t,u,v) = V_0(t,S_1,S_2) = S_1(t)N(d_1) - S_2(t)N(d2) \tag{3.21}$$

where

$$d_1 = \frac{\log\left(\frac{S_1(t)}{S_2(t)}\right) + \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}}, \qquad d_2 = d_1 - \sigma\sqrt{T-t}$$

and

$$\sigma = \sqrt{(\sigma_1 - \sigma_2)^2}$$

In order to have an intuition behind the expression of F_0, let us assume that we want to price the spread option by calculating the expectation of its payoff under the risk neutral measure[4] $\mathbb{Q}$ when $\rho = 1$. In this case, we know that S_1 and S_2 are driven by the same Brownian motion $(W_t)_t$, thus

$$\begin{aligned} V(t,S_1,S_2) &= \mathbb{E}^{\mathbb{Q}}\left[(S_1(T)-S_2(T))^+ \,|\mathcal{F}_t\right] \\ &= \mathbb{E}^{\mathbb{Q}}\left[\left(S_1(t)e^{(r-\frac{\sigma_1^2}{2})(T-t)+\sigma_1(W_T-W_t)} - S_2(t)e^{(r-\frac{\sigma_2^2}{2})(T-t)+\sigma_2(W_T-W_t)}\right)^+ \middle| \mathcal{F}_t\right] \\ &= \mathbb{E}^{\mathbb{Q}}\left[\left(S_1(t)\left(e^{\frac{2\Sigma}{\sigma_1}(r-\frac{\sigma_1^2}{2})(T-t)+2\Sigma(W_T-W_t)}\right)^{\frac{\sigma_1}{2\Sigma}} - S_2(t)\left(e^{\frac{2\Sigma}{\sigma_2}(r-\frac{\sigma_2^2}{2})(T-t)+2\Sigma(W_T-W_t)}\right)^{\frac{\sigma_2}{2\Sigma}}\right)^+ \middle| \mathcal{F}_t\right] \end{aligned}$$

Hence, with some further calculations, we can express the payoff as a weighted difference of the powers of one single process u (which has a volatility of 2Σ). This is what the similarity reduction of the Black&Scholes PDE was all about.

Now that we have found the value of F_0, we can calculate F_1 using equation (3.18). We know that the terminal value is : $F_1(T,u,v) = 0$, hence using *Feynman-Kac generalized formula*, we deduce that

$$F_1(t,u,v) = (t-T)\Sigma^2\left(u^2\frac{\partial^2 F_0}{\partial u^2} - v^2\frac{\partial^2 F_0}{\partial v^2} + u\frac{\partial F_0}{\partial u} - v\frac{\partial F_0}{\partial v}\right) \tag{3.22}$$

We now need to calculate the first and second derivatives of F_0 with respect to u and v.

Proposition 3 *By letting*

$$K_1(t) = e^{\left(\left(\frac{\sigma_1}{2\Sigma}-1\right)r+\sigma_1\left(\frac{\sigma_1}{2}-\Sigma\right)\right)(T-t)} \qquad \text{and} \qquad K_2(t) = e^{\left(\left(\frac{\sigma_2}{2\Sigma}-1\right)r+\sigma_1\left(\frac{\sigma_2}{2}-\Sigma\right)\right)(T-t)}$$

We have the following results (when $\sigma_1 > \sigma_2$[5])

$$\begin{aligned} u\frac{\partial F_0}{\partial u} &= K_1(t)v_t^{\frac{\sigma_1}{2\Sigma}}u_t^{\frac{\sigma_1}{2\Sigma}}\left(\tfrac{\sigma_1}{2\Sigma}N(d_1) + \tfrac{n(d_1)}{2\Sigma\sqrt{T-t}}\right) - K_2(t)v_t^{-\frac{\sigma_2}{2\Sigma}}u_t^{\frac{\sigma_2}{2\Sigma}}\left(\tfrac{\sigma_2}{2\Sigma}N(d_2) + \tfrac{n(d_2)}{2\Sigma\sqrt{T-t}}\right) \\ v\frac{\partial F_0}{\partial v} &= K_1(t)v_t^{\frac{\sigma_1}{2\Sigma}}u_t^{\frac{\sigma_1}{2\Sigma}}\left(\tfrac{\sigma_1}{2\Sigma}N(d_1) + \tfrac{(\sigma_1+\sigma_2)n(d_1)}{2\Sigma(\sigma_1-\sigma_2)\sqrt{T-t}}\right) + K_2(t)v_t^{-\frac{\sigma_2}{2\Sigma}}u_t^{\frac{\sigma_2}{2\Sigma}}\left(\tfrac{\sigma_2}{2\Sigma}N(d_2) - \tfrac{(\sigma_1+\sigma_2)n(d_2)}{2\Sigma(\sigma_1-\sigma_2)\sqrt{T-t}}\right) \\ u^2\frac{\partial^2 F_0}{\partial u^2} &= K_1(t)v_t^{\frac{\sigma_1}{2\Sigma}}u_t^{\frac{\sigma_1}{2\Sigma}}\left(\tfrac{\sigma_1}{2\Sigma}\left(\tfrac{\sigma_1}{2\Sigma}-1\right)N(d_1) + \left(\tfrac{\sigma_1}{\Sigma} - 1 - \tfrac{d_1}{2\Sigma\sqrt{T-t}}\right)\tfrac{n(d_1)}{2\Sigma\sqrt{T-t}}\right) \\ &\quad - K_2(t)v_t^{-\frac{\sigma_2}{2\Sigma}}u_t^{\frac{\sigma_2}{2\Sigma}}\left(\tfrac{\sigma_2}{2\Sigma}\left(\tfrac{\sigma_2}{2\Sigma}-1\right)N(d_2) + \left(\tfrac{\sigma_2}{\Sigma} - 1 - \tfrac{d_2}{2\Sigma\sqrt{T-t}}\right)\tfrac{n(d_2)}{2\Sigma\sqrt{T-t}}\right) \end{aligned}$$

[4] Without using any change of numeraire technique

[5] For the other case we just differentiate with respect to $-d_i$ instead of d_i.

$$v^2\frac{\partial F_0}{\partial v^2} = K_1(t)v_t^{\frac{\sigma_1}{2\Sigma}}u_t^{\frac{\sigma_1}{2\Sigma}}\left(\tfrac{\sigma_1}{2\Sigma}\left(\tfrac{\sigma_1}{2\Sigma}-1\right)N(d_1)+\left(\tfrac{\sigma_1}{\Sigma}-1-\tfrac{(\sigma_1+\sigma_2)d_1}{2\Sigma(\sigma_1-\sigma_2)\sqrt{T-t}}\right)\tfrac{(\sigma_1+\sigma_2)n(d_1)}{2\Sigma(\sigma_1-\sigma_2)\sqrt{T-t}}\right)$$
$$-K_2(t)v_t^{-\frac{\sigma_2}{2\Sigma}}u_t^{\frac{\sigma_2}{2\Sigma}}\left(\tfrac{\sigma_2}{2\Sigma}\left(\tfrac{\sigma_2}{2\Sigma}+1\right)N(d_2)-\left(\tfrac{\sigma_2}{\Sigma}+1+\tfrac{(\sigma_1+\sigma_2)d_2}{2\Sigma(\sigma_1-\sigma_2)\sqrt{T-t}}\right)\tfrac{(\sigma_1+\sigma_2)n(d_2)}{2\Sigma(\sigma_1-\sigma_2)\sqrt{T-t}}\right)$$

where

$$n(x) = \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}$$

The proof of this proposition involves using the usual chain rule differentiation (as when calculating B&S Greeks).

Now that we have found expressions for both F_0 and F_1, let us check the quality of our approximation by comparing it to Margrabe's closed formula for exchange options (i.e Spread Option with strike 0). We will use the following parameters: $S_1(0) = S_2(0) = 100$, $\sigma_1 = 0.4$, $\sigma_2 = 0.1$, $r = 0.05$, $T = 1$ and ρ varying between 0.9 and 1.

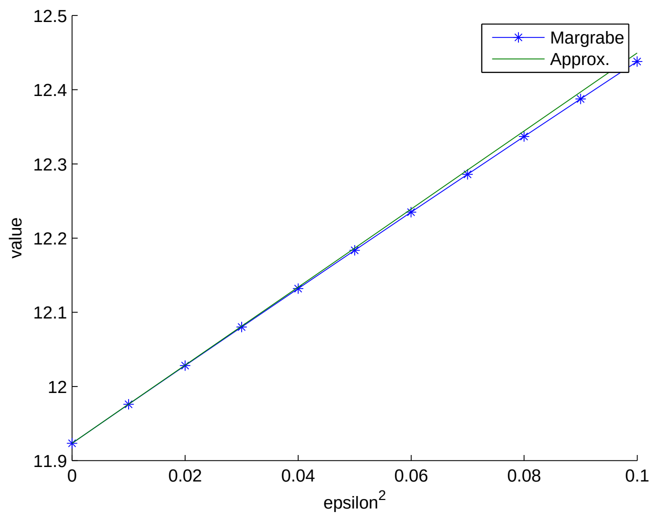

Figure 3.2. Comparison between Margrabe's formula and our approximation for different values of ϵ^2

We can notice from the above figure that the approximation is quite good using the above parameters. Let us now check the behavior of the relative error by changing the volatilities. We can see from figure 3.3 that the relative error tends to explode as the difference between σ_1 and σ_2 gets smaller. However, this error decreases when the difference between the initial values of S_1 and S_2 gets bigger as we can see in figure 3.4.
To understand this behavior, let us analyse Margrabe's formula when $|\sigma_1 - \sigma_2|$ and $|S1 - S2|$ are very small. In this case:

$$\sigma = \sqrt{\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2} = \sqrt{(\sigma_1 - \sigma_2)^2 + 2\sigma_1\sigma_2\epsilon^2} \sim \sqrt{2\sigma_1\sigma_2\epsilon^2}$$

thus

$$d_1 \sim \frac{1}{2}\sqrt{2\sigma_1\sigma_2 T\epsilon^2} \qquad \text{and} \qquad d_2 \sim -d_1$$

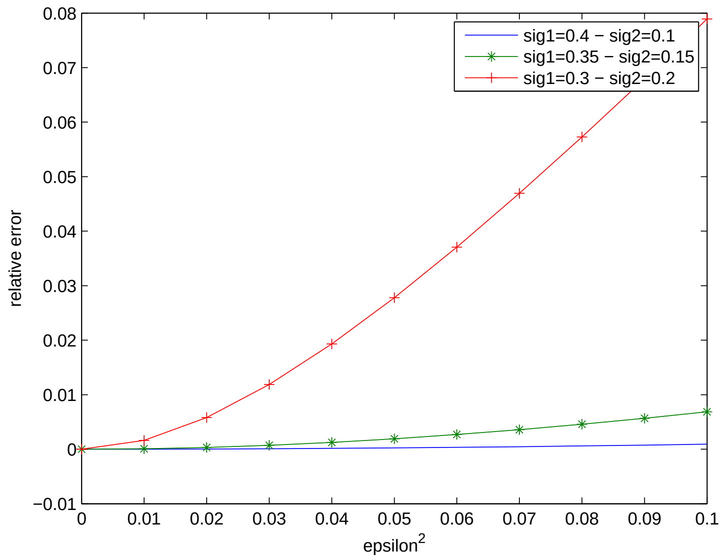

Figure 3.3. relative error between Margrabe's formula and our approximation for different values of σ_1 and σ_2

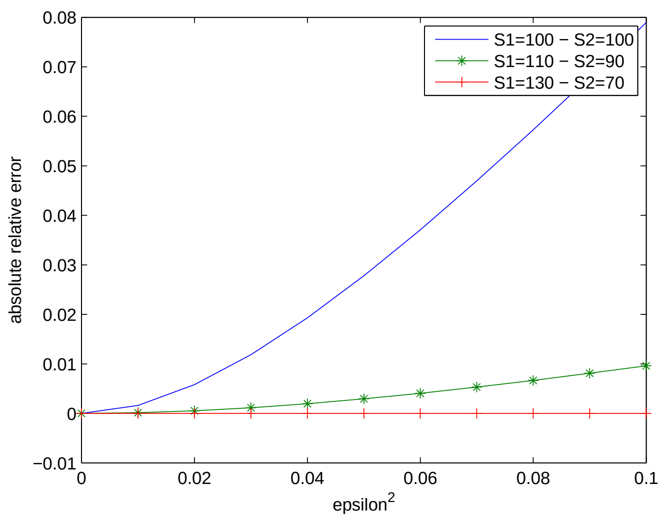

Figure 3.4. relative error between Margrabe's formula and our approximation for different values of S_1 and S_2 in the case $\sigma_1 = 0.3$ and $\sigma_2 = 0.2$

which leads to

$$V(0, S_1, S_2) \sim S_1(0)\left(N(d_1) - N(-d_1)\right) \sim S_1(0)\sqrt{\frac{\sigma_1\sigma_2 T}{\pi}}\epsilon^2$$

hence, in this case, V evolves linearly with $\sqrt{\epsilon^2}$ and not with ϵ^2, which means that our linear approximation fails.

Further more, if we look at the first and second derivatives of F_0 with respect to v, we can see that these involve terms in $\frac{1}{\sigma_1-\sigma_2}$, which means that when $|\sigma_1 - \sigma_2| \ll 1$ these derivatives explode. This makes the error even bigger. We can also check what happens to our formula in the the case where $\sigma_1 = \sigma_2$. In that case if we let[6] $\Sigma = \dfrac{\sigma_1}{2}$ then the terminal condition becomes

$$\begin{aligned} F_0(T,u,v) = \left(u_T v_t - u_T v_t^{-1}\right)^+ &= u_T\left(v_t - v_t^{-1}\right)^+ \\ &= u_T\left(v_t - v_t^{-1}\right)\mathbf{1}_{\{v_t^2>1\}} \end{aligned}$$

We also know that $e^{-r(T-t)}u_T$ is a martingale under the risk neutral measure $\mathbb{Q}$, thus

$$F_0(t,u,v) = e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}\left[u_T\left(v_t - v_t^{-1}\right)\mathbf{1}_{\{v_t^2>1\}}|\mathcal{F}_t\right] = u_t\left(v_t - v_t^{-1}\right)\mathbf{1}_{\{v_t^2>1\}}$$

Now, if we go back to the initial variables S_1 and S_2, we find in that case that

$$u_t = \sqrt{S_1(t)}\sqrt{S_2(t)} \qquad \text{and} \qquad v_t = \sqrt{\frac{S_1(t)}{S_2(t)}}$$

Hence

$$F_0(t,u,v) = V_0(t,S_1,S_2) = (S_1(t) - S_2(t))\mathbf{1}_{\{S_1(t)>S_2(t)\}} \tag{3.23}$$

We can notice that there is a discontinuity in the slope of F_0 when $S_1(t) = S_2(t)$. This can explains the explosion in the error when we take equal initial values for S_1 and S_2 which is due to the discontinuity of the deltas and hence an explosion of the gammas of the option.

Remark 3 *The analysis we have done until now is equivalent to a Taylor expansion on the correlation parameter. Indeed, the Taylor expansion of $V(t,\rho)$ when $\rho = 1-\epsilon^2$ is given by*

$$V(t,\rho) = V(t,1) - \epsilon^2 \left.\frac{\partial V}{\partial \rho}\right|_{\rho=1} + O(\epsilon^4)$$

Also, by differentiating the BSPDE with respect to ρ, we get

$$\mathcal{L}_{BS}^{\rho=1}\left.\frac{\partial V}{\partial \rho}\right|_{\rho=1} = -\sigma_1\sigma_2 S_1 S_2 \left.\frac{\partial^2 V}{\partial S_1 S_2}\right|_{\rho=1}$$

Thus, by Feyman-Kac's formula

$$\left.\frac{\partial V}{\partial \rho}\right|_{\rho=1} = (T-t)\sigma_1\sigma_2 S_1 S_2 \left.\frac{\partial^2 V}{\partial S_1 S_2}\right|_{\rho=1}$$

[6] Recall that the value we assign to Σ does not change the results, it only simplifies the calculations.

We also know that

$$\sigma_1\sigma_2 S_1 S_2 \left.\frac{\partial^2 V}{\partial S_1 S_2}\right|_{\rho=1} = \Sigma^2 u\frac{\partial F}{\partial u} - \Sigma^2 v\frac{\partial F}{\partial v} + \Sigma^2 u^2\frac{\partial^2 F}{\partial u^2} - \Sigma^2 v^2\frac{\partial^2 F}{\partial v^2}$$

thus

$$(T-t)\sigma_1\sigma_2 S_1 S_2 \left.\frac{\partial^2 V}{\partial S_1 S_2}\right|_{\rho=1} = -F_1(t,u,v)$$

Moreover $V(t,1) = F_0(t,u,v)$*, hence*

$$F(t,u,v) = V(t,\rho) = F_0(t,u,v) + \epsilon^2 F_1(t,u,v) + O(\epsilon^4)$$

Consequently, our asymptotic expansion is equivalent to a Taylor expansion on the correlation. Even if the latter is a much simpler approach, our similarity reduction will allow us to give prices for spread options in the general case where $K \neq 0$*.*

3.3 Pricing formula in the general case ($K \neq 0$)

In the previous section, we have studied the case of the exchange option. Let us now deal with the general case of Spread Options with non-zero strike. The first thing to start with, is to analyse the exercise region of F_0 at maturity given in (3.17):

$$\mathcal{D} = \left\{ u^{\frac{\sigma_1}{2\Sigma}} v^{\frac{\sigma_1}{2\Sigma}} - u^{\frac{\sigma_2}{2\Sigma}} v^{-\frac{\sigma_2}{2\Sigma}} > K \right\}$$

3.3.1 Case $\sigma_1 > \sigma_2$

Let us first consider the case where $\sigma_1 > \sigma_2$. If we let $\Sigma = \dfrac{\sigma_1}{2}$ and $\alpha = \dfrac{\sigma_2}{\sigma_1}$, we get the following

$$\mathcal{D} = \left\{ vu - v^{-\alpha}u^{\alpha} > K \right\} \qquad \text{where} \qquad 0 < \alpha < 1$$

Let us consider the two functions f and g defined by

$$f(u) = vu - K, \qquad g(u) = v^{-\alpha}u^{\alpha} \qquad \text{where} \qquad u \in [0, +\infty[$$

The domain $\mathcal{D}$ will then be written

$$\mathcal{D} = \{f(u) > g(u)\}$$

We know that f and g are both increasing functions. Also

$$f(0) = -K < g(0) = 0 \qquad \text{and} \qquad \lim_{u\to+\infty} \frac{g(u)}{f(u)} = 0$$

Thus f and g will necessarily cross at some point that we will denote $u^*(v)$. Figure 3.5 gives an illustration.
$u^*(v)$ can be found numerically using a Newton algorithm. The exercise region will now be

$$\mathcal{D} = \{f(u) > g(u)\} = \{u > u^*(v)\}$$

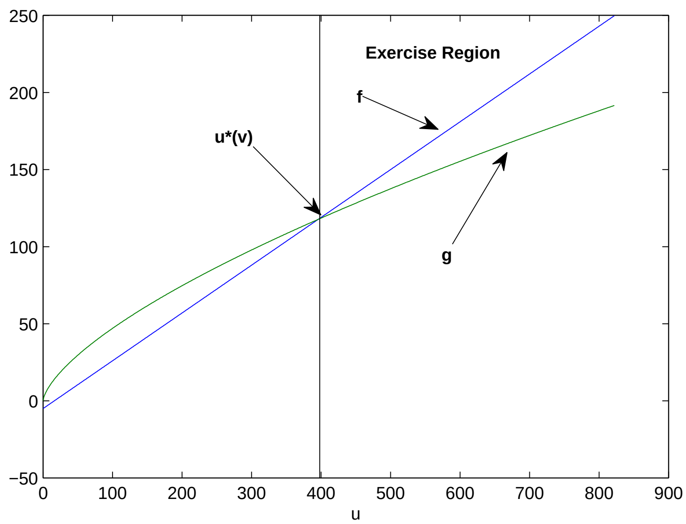

Figure 3.5. Plot of the functions f and g

Let us now go back to the terminal condition of F_0

$$F_0(T,u,v) = \left(u_T v_t - u_T^{\alpha} v_t^{-\alpha} - K\right)^+ = \left(u_T v_t - u_T^{\alpha} v_t^{-\alpha} - K\right) \mathbf{1}_{\left\{u_T v_t - u_T^{\alpha} v_t^{-\alpha} > K\right\}}$$
$$= \left(u_T v_t - u_T^{\alpha} v_t^{-\alpha} - K\right) \mathbf{1}_{\{u_T > u^*(v_t)\}}$$

which leads us to the following proposition:

Proposition 4 *In the case where $\sigma_1 > \sigma_2$, if we let $\alpha = \dfrac{\sigma_2}{\sigma_1}$ and $u^*(v)$ the solution of the equation*

$$vu - v^{-\alpha} u^{\alpha} = K \tag{3.24}$$

then F_0 at time t is given by

$$F_0(t,u,v) = v_t u_t N(d_1) - v_t^{-\alpha} u_t^{\alpha} e^{(\alpha-1)(r+\alpha\frac{\sigma_1^2}{2})(T-t)} N(d_2) - K e^{-r(T-t)} N(d_3) \tag{3.25}$$

where

$$d_1 = \frac{\log\left(\dfrac{u_t}{u^*(v_t)}\right) + \left(r + \frac{\sigma_1^2}{2}\right)(T-t)}{\sigma_1\sqrt{T-t}},$$
$$d_2 = \frac{\log\left(\dfrac{u_t}{u^*(v_t)}\right) + \left(r + (2\alpha-1)\frac{\sigma_1^2}{2}\right)(T-t)}{\sigma_1\sqrt{T-t}},$$
$$d_3 = \frac{\log\left(\dfrac{u_t}{u^*(v_t)}\right) + \left(r - \frac{\sigma_1^2}{2}\right)(T-t)}{\sigma_1\sqrt{T-t}}$$

For the proof, we follow the same steps as in Proposition 2.

Remark 4 *In the special case where* $\alpha = \frac{1}{2}$ *(i.e* $\sigma_1 = 2\sigma_2$*),* $u^*(v)$ *is given explicitly by*

$$u^*(v) = \frac{\left(1 + \sqrt{1 + 4Kv^2}\right)^2}{4v^3}$$

Now that we have found the value of F_0, let us compute F_1. To do so, we need to compute the first and second derivatives of F_0 with respect to u and v. The differentiations with respect to u are straight forward. However, to compute the derivatives with respect to v, we need to differentiate $u^*(v)$. In fact, we can do this by differentiating equation (3.24). We get

$$\begin{aligned}
&\frac{du^*}{dv}(v) = \frac{u^*(v) + \alpha v^{-\alpha-1} u^*(v)^\alpha}{\alpha v^{-\alpha} u^*(v)^{\alpha-1} - v} \\
&\frac{d^2u^*}{dv^2} = \left(\left(\frac{du^*}{dv}(v) - \alpha(\alpha+1) v^{-\alpha-2} u^*(v)^\alpha + \alpha^2 v^{-\alpha-1} u^*(v)^{\alpha-1} \frac{du^*}{dv}(v) \right) \left(\alpha v^{-\alpha} u^*(v)^{\alpha-1} - v \right) \right. \\
&\left. - \left(u^*(v) + \alpha v^{-\alpha-1} u^*(v)^\alpha \right) \left(-\alpha^2 v^{-\alpha-1} u^*(v)^{\alpha-1} + \alpha(\alpha-1) v^{-\alpha} u^*(v)^{-\alpha-2} \frac{du^*}{dv}(v) - 1 \right) \right) \\
&/ \left(\alpha v^{-\alpha} u^*(v)^{\alpha-1} - v \right)^2
\end{aligned} \tag{3.26}$$

where $\alpha v^{-\alpha} u^*(v)^{\alpha-1} - v \neq 0$. We can now compute the derivatives with respect to u and v in order to get F_1.

3.3.2 Case $\sigma_1 < \sigma_2$

If we now consider the case where $\sigma_1 < \sigma_2$ then if we let $\Sigma = \frac{\sigma_2}{2}$ and $\alpha = \frac{\sigma_1}{\sigma_2}$, we get the following domain

$$\mathcal{D} = \left\{ v^\alpha u^\alpha - v^{-1} u > K \right\} \qquad \text{where} \qquad 0 < \alpha < 1$$

If we let the functions f and g now be

$$f(u) = v^{-1}u + K, \qquad g(u) = v^\alpha u^\alpha \qquad \text{where} \qquad u \in [0, +\infty[$$

The domain $\mathcal{D}$ will then be written

$$\mathcal{D} = \{g(u) > f(u)\}$$

We now that the two functions are increasing and

$$f(0) = K > g(0) = 0 \qquad \text{and} \qquad \lim_{u \to +\infty} \frac{g(u)}{f(u)} = 0$$

In this case, we have three possible scenarios:

- f and g never cross

- they cross at one point (tangent)
- they cross at two distinct points

The first two scenarios are straight forward because the payoff will be equal to 0 which implies that the value of the option is 0. In the remaining case, the exercise region of the option will be between the two crossing points, that we will denote $u_1^*(v)$ and $u_2^*(v)$ $(u_1^*(v) < u_2^*(v))$. Here is an illustration:

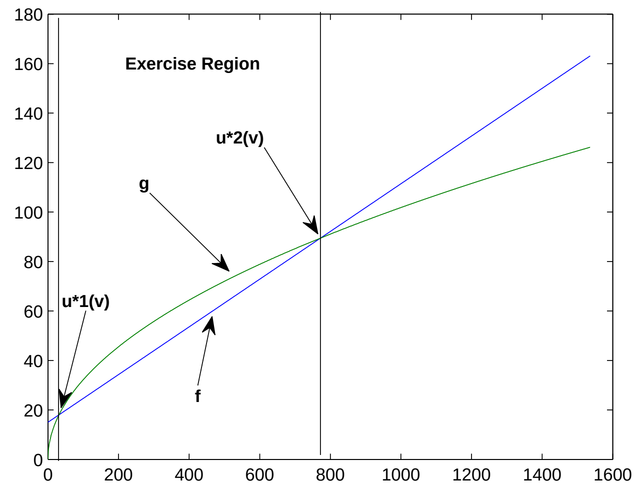

Figure 3.6. Plot of the functions f and g

Thus, in the case where the equation $f(u) = g(u)$ has exactly two distinct solutions $u_1^*(v)$ and $u_2^*(v)$, we have

$$\mathcal{D} = \{g(u) > f(u)\} = \{u_1^*(v) < u < u_2^*(v)\}$$

Let us now go back again to the terminal condition of F_0. In the above case, this will be written:

$$\begin{aligned} F_0(T,u,v) = \left(u_T^\alpha v_t^\alpha - u_T v_t^{-1} - K\right)^+ &= \left(u_T^\alpha v_t^\alpha - u_T v_t^{-1} - K\right) \mathbf{1}_{\left\{u_T^\alpha v_t^\alpha - u_T v_t^{-1} > K\right\}} \\ &= \left(u_T^\alpha v_t^\alpha - u_T v_t^{-1} - K\right) \mathbf{1}_{\left\{u_1^*(v_t) < u_T < u_2^*(v_t)\right\}} \\ &= \left(u_T^\alpha v_t^\alpha - u_T v_t^{-1} - K\right) \left(\mathbf{1}_{\left\{u_T > u_1^*(v_t)\right\}} - \mathbf{1}_{\left\{u_T > u_2^*(v_t)\right\}}\right) \end{aligned}$$

The proposition below then follows:

Proposition 5 *In the case where $\sigma_1 < \sigma_2$, if we let $\alpha = \dfrac{\sigma_1}{\sigma_2}$, then if the equation*

$$v^\alpha u^\alpha - v^{-1} u = K \tag{3.27}$$

has 0 or 1 solutions, then

$$F_0(t,u,v) = 0 \tag{3.28}$$

Otherwise, if (3.27) *has two distinct solutions* $u_1^*(v)$ *and* $u_2^*(v)$ $(u_1^*(v) < u_2^*(v))$*, then* F_0 *at time* t *is given by*

$$F_0(t,u,v) = v_t^\alpha u_t^\alpha e^{(\alpha-1)(r+\alpha\frac{\sigma_2^2}{2})(T-t)}\left(N(d_{1,1}) - N(d_{1,2})\right) - v_t^{-1}u_t\left(N(d_{2,1}) - N(d_{2,2})\right) - Ke^{-r(T-t)}\left(N(d_{3,1}) - N(d_{3,2})\right) \tag{3.29}$$

where

$$d_{1,1} = \frac{\log\left(\frac{u_t}{u_1^*(v_t)}\right) + \left(r + (2\alpha-1)\frac{\sigma_2^2}{2}\right)(T-t)}{\sigma_2\sqrt{T-t}}, \quad d_{1,2} = \frac{\log\left(\frac{u_t}{u_2^*(v_t)}\right) + \left(r + (2\alpha-1)\frac{\sigma_2^2}{2}\right)(T-t)}{\sigma_2\sqrt{T-t}},$$

$$d_{2,1} = \frac{\log\left(\frac{u_t}{u_1^*(v_t)}\right) + \left(r + \frac{\sigma_2^2}{2}\right)(T-t)}{\sigma_2\sqrt{T-t}}, \quad d_{2,2} = \frac{\log\left(\frac{u_t}{u_2^*(v_t)}\right) + \left(r + \frac{\sigma_2^2}{2}\right)(T-t)}{\sigma_2\sqrt{T-t}},$$

$$d_{3,1} = \frac{\log\left(\frac{u_t}{u_1^*(v_t)}\right) + \left(r - \frac{\sigma_2^2}{2}\right)(T-t)}{\sigma_2\sqrt{T-t}}, \quad d_{3,2} = \frac{\log\left(\frac{u_t}{u_2^*(v_t)}\right) + \left(r - \frac{\sigma_2^2}{2}\right)(T-t)}{\sigma_2\sqrt{T-t}}$$

Remark 5 *In the special case where* $\alpha = \frac{1}{2}$ *(i.e* $\sigma_2 = 2\sigma_1$*), if* $v > 2\sqrt{K}$ *then* $u_1^*(v)$ *and* $u_2^*(v)$ *are given explicitly by*

$$u_1^*(v) = \frac{\left(1 - \sqrt{1 - 4Kv^{-2}}\right)^2}{4v^{-3}}, \qquad u_2^*(v) = \frac{\left(1 + \sqrt{1 - 4Kv^{-2}}\right)^2}{4v^{-3}}$$

Here again, to compute F_1, we need to calculate the first and second derivatives of $u_1^*(v)$ and $u_2^*(v)$. We know that these two functions satisfy equation (3.27). Thus, by differentiating we get

$$\frac{du_i^*}{dv}(v) = \frac{\alpha v^{\alpha-1}u_i^*(v)^\alpha + v^{-2}u_i^*(v)}{v^{-1} - \alpha v^\alpha u_i^*(v)^{\alpha-1}}, \qquad i = 1,2 \tag{3.30}$$

where $v^{-1} - \alpha v^\alpha u_i^*(v)^{\alpha-1} \neq 0$. The second derivatives are straight forward when we differentiate the above expression. Once again, we can get an expression for F_1.

3.3.3 Case $\sigma_1 = \sigma_2$

As we have seen in the case when $K = 0$, it can be interesting to check what happens to F_0 when $\sigma_1 = \sigma_2$. First, let us check the terminal condition (by taking $\Sigma = \frac{\sigma_1}{2}$)

$$F_0(T,u,v) = \left(u_T v_t - u_T v_t^{-1} - K\right)^+ = (v_t - v_t^{-1})\left(u_T - \frac{K}{v_t - v_t^{-1}}\right)^+ \mathbf{1}_{\{v_t^2 > 1\}}$$

F_0 in this case can be calculated as a simple Black&Scholes vanilla call. One can also notice the non smoothness when $v_t^2 = 1$, i.e when $S_1(t) = S_2(t)$.

3.4 Comparing our approximation with Kirk's formula

Kirk (1995) introduced an approximation for spread options prices. He considered $X(T) = S_2(T) + K$ to be a lognormal random variable (the correlation between $S_1(T)$ and $X(T)$ is still ρ). Then he applied Margrabe's formula to the exchange option $(S_1(T) - X(T))^+$. He gets the following result:

$$V^K(t, S_1, S_2) = S_1(t)N(d_1^K) - (S_2(t) + Ke^{-r(T-t)})N(d_2^K) \tag{3.31}$$

where

$$d_1^K = \frac{\log\left(\dfrac{S_1(t)}{S_2(t) + Ke^{-r(T-t)}}\right) + \frac{1}{2}\sigma_K^2(T-t)}{\sigma_K\sqrt{T-t}}, \qquad d_2^K = d_1^K - \sigma_K\sqrt{T-t}$$

$$\sigma_K = \sqrt{\sigma_1^2 - 2\rho\sigma_1\sigma_2\frac{S_2(t)}{S_2(t) + Ke^{-r(T-t)}} + \sigma_2^2\left(\frac{S_2(t)}{S_2(t) + Ke^{-r(T-t)}}\right)^2}$$

Let us now see compare our asymptotic approximation with Kirk's formula. For the following, we have used the same set of parameters as in 3.2 with $K = 5$.

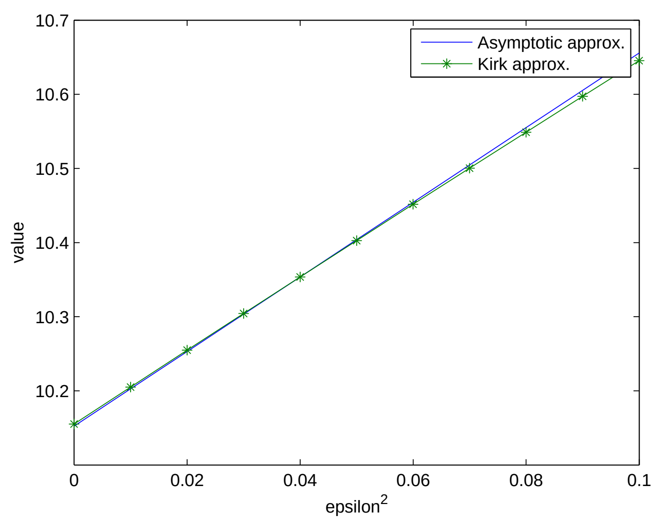

Figure 3.7. Comparison between Kirk's formula and our approximation for different values of ϵ^2

We can see from the figure that both approximations match when the correlation is close to 1. However, as we have seen in 3.2, our approximation becomes less accurate when σ_1 and σ_2 are very close as we can see on the following figure.

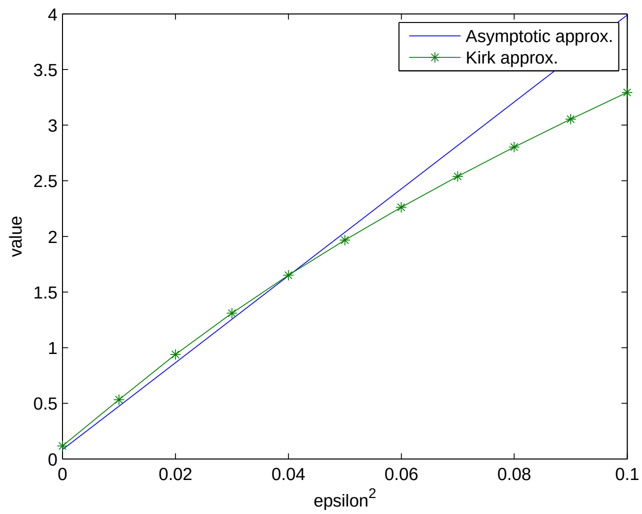

Figure 3.8. Comparison between Kirk's formula and our approximation for $\sigma_1 = 0.3$ and $\sigma_2 = 0.28$

This explosion of the error is again due to the non smoothness in $S_1(t) = S_2(t)$ when $|\sigma_1 - \sigma_2| \ll 1$.

4 Pricing by approximating the bivariate Normal distribution

In this section, we are going to propose an alternative method to price spread options. In fact, we are going to compute the expectation of the payoff by approximating the bivariate normal distribution when the correlation is close to 1. If we let

$$\begin{aligned} a_i &= \log(S_i(t)) - \frac{\sigma_i^2}{2}(T-t) \\ b_i &= \sigma_i \sqrt{T-t} \end{aligned}$$

for $i = 1, 2$, then the value of the spread option with strike K at time t is given by

$$V_t = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \left(e^{a_1 + b_1 x_1} - e^{a_2 + b_2 x_1} - K e^{-r(T-t)} \right)^+ h(x_1, x_2) dx_1 dx_2 \qquad (4.32)$$

where

$$h(x_1, x_2) = \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left(-\frac{x_1^2 - 2\rho x_1 x_2 + x_2^2}{2(1-\rho^2)} \right) \qquad (4.33)$$

If we plot the contour of h is the (x_1, x_2) plane, we get

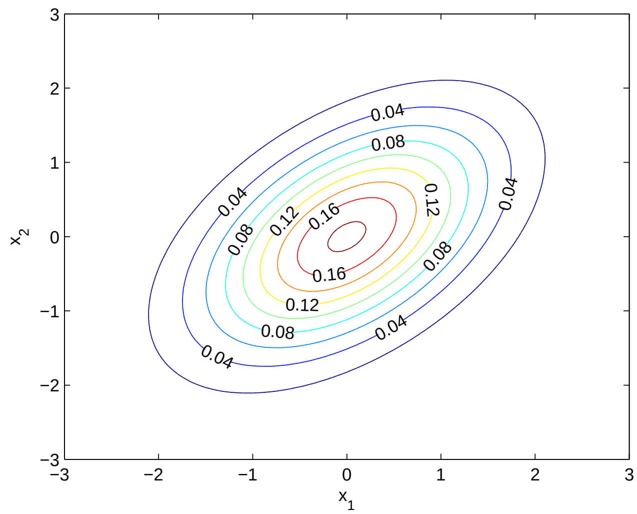

Figure 4.9. Contour of the bivariate normal distribution ($\rho = 0.5$)

We want to find an approximation of this bivariate normal distribution when the correlation is close to 1. If we let $\rho = 1 - \epsilon^2$ then h will be given by

$$h(x_1, x_2) = \frac{1}{2\pi\epsilon\sqrt{2-\epsilon^2}} \exp\left(-\frac{(x_1 - x_2)^2 + 2\epsilon^2 x_1 x_2}{2\epsilon^2(2-\epsilon^2)}\right) \tag{4.34}$$

Let us first plot its contour for some large enough ρ

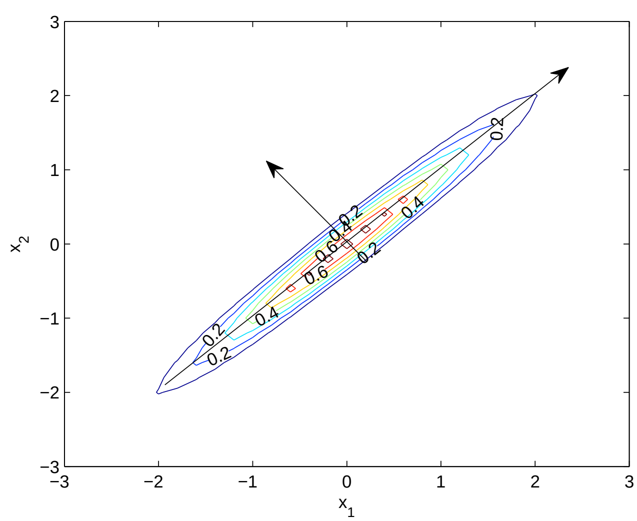

Figure 4.10. Contour of the bivariate normal distribution ($\rho = 0.98$)

In this case, we can notice that the variations around the perpendicular to the ellipse are very small. Thus, we can first start by rotating the referential in order to match the axes of the ellipse. The rotation angle θ is equal to $\frac{\pi}{4}$, thus, if let (ξ, η) be the new referential, we get

$$\begin{cases} \xi = \cos(\theta)x_1 + \sin(\theta)x_2 = (x_1 + x_2)/\sqrt{2} \\ \eta = -\sin(\theta)x_1 + \cos(\theta)x_2 = (-x_1 + x_2)/\sqrt{2} \end{cases} \tag{4.35}$$

Thus,

$$h(x_1, x_2) = f_\epsilon(\xi)g_\epsilon(\eta) \tag{4.36}$$

where

$$f_\epsilon(\xi) = \frac{1}{\sqrt{2\pi(2-\epsilon^2)}} \exp\left(-\frac{\xi^2}{2(2-\epsilon^2)}\right) \tag{4.37}$$

and

$$g_\epsilon(\eta) = \frac{1}{\sqrt{2\pi}\epsilon} \exp\left(-\frac{\eta^2}{2\epsilon^2}\right) \tag{4.38}$$

And the payoff function is given by:

$$F(\xi, \eta) = \left(e^{a_1 + \frac{\sqrt{2}}{2}b_1(\xi-\eta)} - e^{a_2 + \frac{\sqrt{2}}{2}b_2(\xi+\eta)} - Ke^{-r(T-t)}\right)^+ \tag{4.39}$$

Hence, the value of the spread option is given by

$$V_t^\epsilon = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} F(\xi, \eta)f_\epsilon(\xi)g_\epsilon(\eta)d\xi d\eta \tag{4.40}$$

Let us now see what happens to this formula as $\epsilon \to 0$. First of all, $f_\epsilon(\xi)$ becomes $f_0(\xi)$ which the density function of a normal distribution $N(0, 2)$. However, the behavior of $g_\epsilon(\eta)$ is more interesting. Indeed

$$g_\epsilon(\eta) \xrightarrow[\epsilon \to 0]{} \delta(\eta) \tag{4.41}$$

where δ is the Dirac delta function. Thus if we let

$$G_\epsilon(\eta) = \int_{-\infty}^{+\infty} F(\xi, \eta)f_\epsilon(\xi)d\xi \tag{4.42}$$

then we can write

$$V_t^\epsilon = \int_{-\infty}^{+\infty} G_\epsilon(\eta)g_\epsilon(\eta)d\eta \tag{4.43}$$

and by noticing that the function G_0 is smooth enough

$$V_t^\epsilon \xrightarrow[\epsilon \to 0]{} V_t^0 = \int_{-\infty}^{+\infty} G_0(\eta)\delta(\eta)d\eta = G_0(0) \tag{4.44}$$

Hence,

$$
\begin{aligned}
V_t^0 &= \int_{-\infty}^{+\infty} F(\xi,0) f_0(\xi) d\xi \\
&= \int_{-\infty}^{+\infty} \left(e^{a_1+\frac{\sqrt{2}}{2}b_1\xi} - e^{a_2+\frac{\sqrt{2}}{2}b_2\xi} - Ke^{-r(T-t)}\right)^+ \frac{1}{\sqrt{2\pi}\sqrt{2}} e^{-\frac{\xi^2}{4}} d\xi \\
&= \int_{-\infty}^{+\infty} \left(e^{a_1+b_1 s} - e^{a_2+b_2 s} - Ke^{-r(T-t)}\right)^+ \frac{1}{\sqrt{2\pi}} e^{-\frac{s^2}{2}} ds \qquad (s = \frac{\sqrt{2}}{2}\xi)
\end{aligned}
$$

which is the expectation of the payoff of the spread option in the case $\rho = 1$. Therefore, $V_t^0 = F_0(t,u,v)$ (where F_0 has been defined in the previous section for $K \neq 0$). Let us now try to approximate V_t^ϵ for some small ϵ by putting $\eta = \epsilon u$. In that case we get

$$
\begin{aligned}
V_t^\epsilon &= \int_{-\infty}^{+\infty} G_\epsilon(\epsilon u) g_\epsilon(\epsilon u) \epsilon du && (4.45)\\
&= \int_{-\infty}^{+\infty} G_\epsilon(\epsilon u) \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{u^2}{2}\right) du && (4.46)
\end{aligned}
$$

Also, for a small ϵ we can write:

$$
G_\epsilon(\epsilon u) = G_\epsilon(0) + \epsilon u G_\epsilon'(0) + \frac{1}{2}\epsilon^2 u^2 G_\epsilon''(0) + O(\epsilon^3) \tag{4.47}
$$

Thus,

$$
V_t^\epsilon = G_\epsilon(0) + \epsilon G_\epsilon'(0) \int_{-\infty}^{+\infty} u \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}} du + \frac{1}{2}\epsilon^2 G_\epsilon''(0) \int_{-\infty}^{+\infty} u^2 \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}} du + O(\epsilon^3) \tag{4.48}
$$

$$
= G_\epsilon(0) + \frac{1}{2} G_\epsilon''(0)\epsilon^2 + O(\epsilon^3) \tag{4.49}
$$

Let us now compute $G_\epsilon''(\eta)$.

$$
G_\epsilon''(\eta) = \int_{-\infty}^{+\infty} \frac{\partial^2 F(\xi,\eta)}{\partial \eta^2} f_\epsilon(\xi) d\xi \tag{4.50}
$$

If we let

$$
\mathcal{D}(\xi,\eta) = \left\{ e^{a_1+\frac{\sqrt{2}}{2}b_1(\xi-\eta)} - e^{a_2+\frac{\sqrt{2}}{2}b_2(\xi+\eta)} > Ke^{-r(T-t)} \right\} \tag{4.51}
$$

Then

$$
G_\epsilon''(\eta) = \int_{-\infty}^{+\infty} \left(\frac{1}{2} b_1^2 e^{a_1+\frac{\sqrt{2}}{2}b_1(\xi-\eta)} - \frac{1}{2} b_2^2 e^{a_2+\frac{\sqrt{2}}{2}b_2(\xi+\eta)} \right) \mathbf{1}_{\mathcal{D}(\xi,\eta)} f_\epsilon(\xi) d\xi \tag{4.52}
$$

Thus,

$$G''_\epsilon(0) = \int_{-\infty}^{+\infty} \left(\frac{1}{2} b_1^2 e^{a_1 + \frac{\sqrt{2}}{2} b_1 \xi} - \frac{1}{2} b_2^2 e^{a_2 + \frac{\sqrt{2}}{2} b_2 \xi} \right) \mathbf{1}_{\mathcal{D}(\xi,0)} f_\epsilon(\xi) d\xi \tag{4.53}$$

Hence, in order to get our approximation, we just need to find simpler expressions $G_\epsilon(0)$ and $G''_\epsilon(0)$. To do so, we just need to perform the change of variable $s = \frac{\xi}{\sqrt{2-\epsilon^2}}$ in order to integrate on the standard normal distribution. Then, we can integrate using the results from the previous section (i.e when we introduced this special type of power options).

Remark 6 *One important remark here is that using this probabilistic approach, the ϵ term is part of the calculations (i.e $G_\epsilon(0)$ and $G''_\epsilon(0)$ depend on ϵ), and we use the fact that it is small to get an approximation of the price. However, in the case where we used the PDE technique, we first approximated the result, then we have calculated F_0 and F_1 independently of ϵ.*

5 Generalization to a spread option on three assets

In this section, we are going to explain how we can generalize our previous results to the case where we have a spread option written on three assets. This can for example be a 3:2:1 crack spread which has a payoff of the form

$$V(T, S_1, S_2, S_3) = (S_1(T) + S_2(T) - S_3(T) - K)^+ \tag{5.54}$$

In that case, V satisfies the three dimensional Black & Scholes PDE

$$\frac{\partial V}{\partial t} + \frac{1}{2}\left(\sigma_1^2 S_1^2 \frac{\partial^2 V}{\partial S_1^2} + + \sigma_2^2 S_2^2 \frac{\partial^2 V}{\partial S_2^2} + \sigma_3^2 S_3^2 \frac{\partial^2 V}{\partial S_3^2} + 2\rho_{1,2}\sigma_1\sigma_2 S_1 S_2 \frac{\partial^2 V}{\partial S_1 \partial S_2} + 2\rho_{1,3}\sigma_1\sigma_3 S_1 S_3 \frac{\partial^2 V}{\partial S_1 \partial S_3} \right.$$
$$\left. + 2\rho_{2,3}\sigma_2\sigma_3 S_2 S_3 \frac{\partial^2 V}{\partial S_2 \partial S_3} \right) + r\left(S_1 \frac{\partial V}{\partial S_1} + S_2 \frac{\partial V}{\partial S_2} + S_3 \frac{\partial V}{\partial S_3} \right) - rV = 0$$
$$V(T, S_1, S_2, S_3) = (S_1(T) + S_2(T) - S_3(T) - K)^+ \tag{5.55}$$

The correlation matrix is given in the form

$$\mathcal{C} = \begin{pmatrix} 1 & \rho_{1,2} & \rho_{1,3} \\ \rho_{1,2} & 1 & \rho_{2,3} \\ \rho_{1,3} & \rho_{2,3} & 1 \end{pmatrix} \tag{5.56}$$

We will suppose that the three assets are highly correlated. If we let ϵ such that $\epsilon^2 \ll 1$, we can write $\rho_{1,2} = 1 - a\epsilon^2$, $\rho_{1,3} = 1 - b\epsilon^2$ and $\rho_{2,3} = 1 - c\epsilon^2$ for some constants a, b, c. In this case, $\mathcal{C}$ becomes

$$\mathcal{C} = \begin{pmatrix} 1 & 1 - a\epsilon^2 & 1 - b\epsilon^2 \\ 1 - a\epsilon^2 & 1 & 1 - c\epsilon^2 \\ 1 - b\epsilon^2 & 1 - c\epsilon^2 & 1 \end{pmatrix} \tag{5.57}$$

Our aim is to diagonalize this matrix. Let us first check if this matrix is positive definite. We can notice that the upper left 1-by-1 and upper left 2-by-2 corners of $\mathcal{C}$ have strictly positive determinants. Thus, according to Sylvester criterion, we just need to check if the determinant of $\mathcal{C}$ is strictly positive. This is given by

$$\det(\mathcal{C}) = \left(a^2 + b^2 + c^2 - (a-b)^2 - (a-c)^2 - (b-c)^2\right)\epsilon^4 - 2abc\epsilon^6 \tag{5.58}$$

For a,b,c close enough and a sufficiently small ϵ^2, $\det(\mathcal{C}) > 0$ and $\mathcal{C}$ is positive definite. Let us now try to find the eigenvalues of $\mathcal{C}$. These solve the following equation for λ

$$\det(\mathcal{C} - \lambda I) = 0 \tag{5.59}$$

where I is the identity matrix. This is equivalent to

$$-\lambda^3 + 3\lambda^2 + \left((a^2+b^2+c^2)\epsilon^4 - 2(a+b+c)\epsilon^2\right)\lambda + (a^2+b^2+c^2 - (a-b)^2 - (a-c)^2 - (b-c)^2)\epsilon^4 - 2abc\epsilon^6 = 0$$

If we let $\lambda \sim \lambda_0 + \epsilon^2\lambda_1 + ...$, then we get

$$\lambda_0^3 - 3\lambda_0^2 = 0$$

Then,

$$\lambda_0 = 3 \qquad \text{or} \qquad \lambda_0 = 0$$

Which means that $\lambda = 3 + O(\epsilon^2)$ is one of the eigenvalues. Let us now find the other ones. If we let $\lambda = \epsilon^2\mu$ then

$$-\mu^3\epsilon^6 + 3\mu^2\epsilon^4 + \left((a^2+b^2+c^2)\epsilon^4 - 2(a+b+c)\epsilon^2\right)\mu\epsilon^2 + (a^2+b^2+c^2 - (a-b)^2 - (a-c)^2 - (b-c)^2)\epsilon^4 - 2abc\epsilon^6 = 0$$

which can be simplified to

$$-\mu^3\epsilon^2 + 3\mu^2 + \left((a^2+b^2+c^2)\epsilon^2 - 2(a+b+c)\right)\mu + (a^2+b^2+c^2 - (a-b)^2 - (a-c)^2 - (b-c)^2) - 2abc\epsilon^2 = 0$$

If we now let $\mu \sim \mu_0 + \epsilon^2\mu_1 + ...$, then

$$3\mu_0^2 - 2(a+b+c)\mu_0 + a^2 + b^2 + c^2 - (a-b)^2 - (a-c)^2 - (b-c)^2 = 0$$

The discriminant of this equation is given by:

$$\Delta = 16\left(a^2 + b^2 + c^2 - ab - ac - bc\right)$$

which is strictly positive if we choose a,b and c such that $a^2 + b^2 + c^2 > ab + ac + bc$. Thus

$$\mu_0^+ = \frac{(a+b+c) + 2\sqrt{a^2+b^2+c^2-ab-ac-bc}}{3} \tag{5.60}$$

and

$$\mu_0^- = \frac{(a+b+c) - 2\sqrt{a^2+b^2+c^2-ab-ac-bc}}{3} \tag{5.61}$$

Hence, the other eigenvalues are given by $\lambda = \mu_0^+\epsilon^2 + O(\epsilon^4)$ and $\lambda = \mu_0^-\epsilon^2 + O(\epsilon^4)$. We can carry on applying the same technique to get the remaining terms of the eigenvalues. Let us note these eigenvalues λ^1, λ^2 and λ^3 where $\lambda^1 > \lambda^2 > \lambda^3$. As in the 2 dimensional case, we can find a vector (u, v, w) such that $V(t, S_1, S_2, S_3) = F(t, u, v, w)$ and F follows the PDE (for some Σ as in the 2d case)

$$\frac{\partial F}{\partial t} + \lambda^1\Sigma^2 u^2\frac{\partial^2 F}{\partial u^2} + \lambda^2\Sigma^2 v^2\frac{\partial^2 F}{\partial v^2} + \lambda^3\Sigma^2 w^2\frac{\partial^2 F}{\partial w^2} + g(u\frac{\partial F}{\partial u}, v\frac{\partial F}{\partial v}, w\frac{\partial F}{\partial w}) - rF = 0 \tag{5.62}$$

Here again, if we consider the following expansion

$$F(t,u,v,w) \sim F_0(t,u,v,w) + \epsilon^2 F_1(t,u,v,w) + \epsilon^4 F_2(t,u,v,w)... \tag{5.63}$$

then F_0 will satisfy

$$\frac{\partial F_0}{\partial t} + 3\Sigma^2 u^2\frac{\partial^2 F_0}{\partial u^2} + ru\frac{\partial F_0}{\partial u} - rF_0 = 0 \tag{5.64}$$

with some suitable terminal condition which will be a difference between three power options similar to the ones in the 2d case, and u will be the only variable. Then we can solve for F_1 and F_2 using again *Feyman-Kac generalized formula* after calculating the deltas and gammas with respect to u, v and w.

Remark 7 *The result above can be generalized to a spread option on n assets. In fact, if the correlations in the correlation matrix $\mathcal{C}$ are all close to 1, then the leading order value of the highest eigenvalue will be equal to n. Indeed, if λ^1,...,λ^n ($\lambda^1 > ... > \lambda^n$)are the eigenvalues of $\mathcal{C}$, then we know that*

$$\sum_{i=1}^{n}\lambda^i = Tr(\mathcal{C}) = n \qquad \text{and} \qquad \prod_{i=1}^{n}\lambda^i = \det(\mathcal{C}) = O(\epsilon^{2(n-1)}) \tag{5.65}$$

which means that $\lambda^1 = n + O(\epsilon^2)$. Also, the eigenvector associated to this value will be $(1, 1, ..., 1)$. Thus, we can again find $(u_1, ..., u_n)$ such that $F(t, u_1, ..., u_n) = V(t, u_1, ..., u_n)$ and F satisfies the PDE

$$\frac{\partial F}{\partial t} + \Sigma^2\sum_{i=1}^{n}\lambda^i u_i^2\frac{\partial^2 F}{\partial u_i^2} + g(u_1\frac{\partial F}{\partial u_1}, ..., u_n\frac{\partial F}{\partial u_n}) - rF = 0 \tag{5.66}$$

and again we can consider the expansion

$$F(t,u_1,...,u_n) \sim F_0(t,u_1,...,u_n) + \epsilon^2 F_1(t,u_1,...,u_n) + ... + \epsilon^{2(n-1)}F_{n-1}(t,u_1,...,u_n)... \tag{5.67}$$

then F_0 will satisfy

$$\frac{\partial F_0}{\partial t} + n\Sigma^2 u_1^2\frac{\partial^2 F_0}{\partial u_1^2} + ru_1\frac{\partial F_0}{\partial u_1} - rF_0 = 0 \tag{5.68}$$

for some suitable terminal condition where u_1 is the only variable. And then we carry on with the same technique as earlier to get $F_1, ..., F_{n-1}$.

6 Conclusion

Throughout this study, we have seen that Matched Asymptotic Expansions techniques can be very useful in order to get systematic approximations for spread options prices written on two or more assets that are very correlated (crack spreads for example). We have also seen that our approximation is quite good a part from some critical regions, especially when the volatilities of the assets are very close. We have first simplified our framework to the case of exchange options in order to check the accuracy of our approximation compared to Margrabe's formula (which is an exact formula), then we have used this approximation to price spread options in the general case. We have also described an alternative approach to price spread options by approximating the bivariate normal distribution when the correlation is close to 1. Finally, we have showed how we can generalize the previous results to the case where we have more than two assets. This approach can also be extended to a different types of multi-assets options (Basket options, Rainbow options...). In that case, we just need to be careful with the terminal condition and make sure that we can find a way to express it as a vanilla option on a single asset. However, the approximation is only valid for highly correlated assets, thus before using it, we need to make sure that the correlation will not drop in the future. Still, we can see that using these Perturbation techniques, we have managed to find formulas that are easily implemented and less time consuming compared to the usual numerical techniques.

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