The Real Field with an Irrational Power Function and a Dense Multiplicative Subgroup

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Abstract

In recent years the field of real numbers expanded by a multiplicative subgroup has been studied extensively. In this thesis, the known results will be extended to expansions of the real field. I will consider the structure $\tilde{\mathbb{R}}$ consisting of the field of real numbers and an irrational power function.

Using Schanuel conditions, I will give a first-order axiomatization of expansions of $\tilde{\mathbb{R}}$ by a dense multiplicative subgroup which is a subset of the real algebraic numbers. It will be shown that every definable set in such a structure is a boolean combination of existentially definable sets and that these structures have o-minimal open core. A proof will be given that the Schanuel conditions used in proving these statements hold for co-countably many real numbers.

The results mentioned above will also be established for expansions of $\tilde{\mathbb{R}}$ by dense multiplicative subgroups which are closed under all power functions definable in $\tilde{\mathbb{R}}$. In this case the results hold under the assumption that the Conjecture on intersection with tori is true.

Finally, the structure consisting of $\tilde{\mathbb{R}}$ and the discrete multiplicative subgroup $2^{\mathbb{Z}}$ will be analyzed. It will be shown that this structure is not model complete. Further I develop a connection between the theory of Diophantine approximation and this structure.

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Contents

Chapter 1

Introduction

In this introduction we start by discussing the context of this thesis. Then all results of this thesis will be stated and it will be explained how they fit into the broader perspective. Finally the structure of this work will be explained.

1.1 Introduction

Let $\overline{\mathbb{R}} = (\mathbb{R}, <, +, \cdot, 0, 1)$ be the field of real numbers. In [vdD85] van den Dries described the theory of $(\overline{\mathbb{R}}, 2^{\mathbb{Z}})$ and showed that this theory is near model complete, ie. that every formula is equivalent to a boolean combination of existential formulas. The main tool in his proof was a valuation-theoretic result known as the valuation inequality. Van den Dries himself generalized this valuation-theoretic machinery in [vdD97] to arbitrary power-bounded o-minimal expansions of a real closed field. This allowed Miller to extend van den Dries' results on $(\overline{\mathbb{R}}, 2^{\mathbb{Z}})$ to certain expansions of $\overline{\mathbb{R}}$. Miller used van den Dries' valuation-theoretic method to give an axiomatization for structures consisting of an o-minimal expansion $\mathfrak{R}$ of $\overline{\mathbb{R}}$ which is polynomially bounded with field of exponents $\mathbb{Q}$, and the group $2^{\mathbb{Z}}$. In [M05-2] he also proved that the theory of this structure is d-minimal, which means that every definable set in any model of the theory is the union of an open set and finitely many discrete sets. Miller's work directly implies the following corollary.

Theorem 1.1.1 *Let $\mathfrak{R}$ be a model-complete, o-minimal expansion of $\overline{\mathbb{R}}$, which is polynomially bounded with field of exponents $\mathbb{Q}$. Then $(\mathfrak{R}, 2^{\mathbb{Z}})$ is model-complete.*

These results were then extended by Friedman and Miller to expansions of o-minimal structures by fast sequences in [FM05] and by Miller and Tyne to expansions of o-minimal structures by iterating sequences in [MT06].

Another line of research was started by van den Dries in [vdD98-2], where dense pairs of o-minimal structures were investigated. Given an o-minimal theory T extending the theory of real closed fields, and two models A, B of T, where A is dense in B, let $\mathfrak{L}_T$ be the language of T and $\mathfrak{L}_T(U)$ obtained from $\mathfrak{L}_T$ by adding a unary relation symbol U, and consider (B, A) as an $\mathfrak{L}_T(U)$-structure. Van den Dries showed that

Theorem 1.1.2 *Every subset of B^m definable in (B, A) is a boolean combination of subsets of B^m defined by*

$$\exists y_1 ... \exists y_n \; U(y_1) \wedge ... \wedge U(y_n) \wedge \varphi(x_1, ..., x_m, y_1, ..., y_n),$$

where φ is an $\mathfrak{L}_T$-formula.

Let B be a model of an arbitrary theory and U a subset of B. We will say that (B, U) is *near model complete*, if it satisfies the conclusion of the above theorem.
In [vdDG06] van den Dries and Günaydin succeeded in relaxing the assumptions on A in Theorem 1.1.2. They started to analyze the structure consisting of the real field $\overline{\mathbb{R}}$ and a dense multiplicative subgroup Γ. They gave an axiomatization of the structure $(\overline{\mathbb{R}}, \Gamma)$ in the case that Γ has the Mann property. The Mann property was introduced in the model-theoretic study of structures consisting of a field and a multiplicative subgroup by Zilber in [Z90]. Zilber used this tool to show that the complex numbers with a predicate for a subgroup of the roots of unity is ω-stable. Further in [Z03] he proved with the help of the Mann property that the real field with a predicate for the complex roots of unity is near model complete. The Mann property is defined as follows: Let G be any subgroup of the multiplicative group $\mathbb{R}^\times$. Consider equations of the form

$$a_1x_1 + ... + a_nx_n = 1,$$

where $a_1, ..., a_n \in \mathbb{Q}$. We say a solution $(b_1, ..., b_n) \in G^n$ is *non-degenerate* if for every non-empty subset I of $\{1, ..., n\}$ $\sum_{i \in I} a_i b_i \neq 0$. Then G has the *Mann property* if every equation of the above type has only finitely many non-degenerate solutions in G^n. Examples of groups satisfying the Mann property are all subgroups of $\mathbb{R}^\times$, which are in the divisible closure of a finitely generated subgroup of $\mathbb{R}^\times$. So in particular $2^{\mathbb{Q}}$ and $2^{\mathbb{Z}}3^{\mathbb{Z}}$. They considered the case of a dense subgroup Γ of the multiplicative subgroup $\mathbb{R}^\times$, which satisfies the Mann property and for which the group $\Gamma/\Gamma^{[n]}$ is finite for every $n \in \mathbb{N}$. Note that $\Gamma^{[n]}$ is the subgroup $\{g^n | g \in \Gamma\}$ and not the n-th cartesian power of Γ. Let $\mathfrak{L}$ be the language of $\overline{\mathbb{R}}$ and let $\mathfrak{L}(G)$ be the language obtained from $\mathfrak{L}$ by adding a unary relation symbol G.

Theorem 1.1.3 *Let Γ be a multiplicative subgroup of $\mathbb{R}$ satisfying the Mann property. If $\Gamma/\Gamma^{[n]}$ is finite for all $n \in \mathbb{N}$, then every subset of $\mathbb{R}^m$ definable in $(\overline{\mathbb{R}}, \Gamma)$ is a boolean combination of subsets of $\mathbb{R}^m$ defined in $(\overline{\mathbb{R}}, \Gamma)$ by*

$$\exists y_1 ... \exists y_n \; G(y_1) \wedge ... \wedge G(y_n) \wedge \varphi(x_1, ..., x_m, y_1, ..., y_n),$$

where φ is a quantifier-free $\mathfrak{L}$-formula.

Independently, Belegradek and Zilber showed in [BZ08] that Theorem 1.1.3 also holds for the real field augmented by a binary relation which is a subgroup of $\mathbb{C}^\times$ contained in the the divisible hull of a finitely generated subgroup of the unit circle.

The work of this thesis also fits into a research program of Miller. The aim is to decide which expansions of o-minimal structures defines the set of integers and which have o-minimal open core (see [M05-2], [M05], [DMS08]). In [MS99], Miller and Speissegger asked the question whether $(\overline{\mathbb{R}}, 2^{\mathbb{Z}}3^{\mathbb{Z}})$ has o-minimal open core and whether $(\overline{\mathbb{R}}, 2^{\mathbb{Z}}, 3^{\mathbb{Z}})$ defines the integers. While the first question was answered positively by Berenstein, Ealy and Günaydin in [BEG07], there is no known solution for the second question. Not much is known in the second case except that Günaydin shows in [G08] that $(\overline{\mathbb{R}}, 2^{\mathbb{Z}}3^{\mathbb{Z}}, 2^{\mathbb{Z}})$ is near model complete. Note further that Günaydin also shows that the group $3^{\mathbb{Z}}$ is *not* definable in $(\overline{\mathbb{R}}, 2^{\mathbb{Z}}3^{\mathbb{Z}}, 2^{\mathbb{Z}})$.

1.2 Summary

In this thesis, we extend some of the above results. The main question considered is whether Theorem 1.1.3 can be extended to o-minimal expansions of the real field. The proof of Theorem 1.1.3 is algebraic and the main problem is to extend this machinery to analytic situations. This does not work in every situation. Consider for example case of the real field with restricted analytic functions $\mathbb{R}_{an}$ (see Chapter 2 Section 2 for a precise definition of $\mathbb{R}_{an}$). Note that $\mathbb{R}_{an}$ satisfies all assumptions of Theorem 1.1.1, and hence $(\mathbb{R}_{an}, 2^{\mathbb{Z}})$ is model-complete. But $(\mathbb{R}_{an}, 2^{\mathbb{Q}})$ is not tame at all:

Theorem 1.2.1 $(\mathbb{R}_{an}, 2^{\mathbb{Q}})$ *defines the integers.*

<u>Proof:</u> First note that the function $log_2|_{(2,4)}$ is definable in $\mathbb{R}_{an}$. Thus the set $\mathbb{Q}\cap(1,2)$ is definable in $(\mathbb{R}_{an}, 2^{\mathbb{Q}})$ and hence $\mathbb{Q}$ is definable in $(\mathbb{R}_{an}, 2^{\mathbb{Q}})$. But by [R59], $(\mathbb{Q}, +, \cdot)$ already defines $\mathbb{Z}$.

□

This result suggests that we need to consider structures which not even locally define the logarithm. In this thesis we restrict ourselves to structures of the form $\tilde{\mathbb{R}} := (\mathbb{R}, +, \cdot, 0, 1, x \mapsto \begin{cases} x^\tau, & x > 0, \\ 0, & x \leq 0. \end{cases})$, where $\tau \in \mathbb{R} - \mathbb{Q}$. This structure is o-minimal and polynomially-bounded with field of exponents $\mathbb{Q}(\tau)$. The main result of this thesis will be to show that for $\tilde{\mathbb{R}}$ and certain dense multiplicative subgroups Theorem 1.1.3 holds.

We will consider two different classes of multiplicative subgroups: the subgroups which are orthogonal to the new power functions, and subgroups which are closed under all definable power functions. Consider for example, the subgroups $2^{\mathbb{Q}}$ and

$2^{\mathbb{Q}(\tau)}$. The difference between $2^{\mathbb{Q}}$ and $2^{\mathbb{Q}(\tau)}$ is that $2^{\mathbb{Q}(\tau)}$ is closed under all power functions definable in $\tilde{\mathbb{R}}$, ie. for all $g \in 2^{\mathbb{Q}(\tau)}$ and all $p \in \mathbb{Q}(\tau)$ we have that $g^p \in 2^{\mathbb{Q}(\tau)}$. In contrast, $2^{\mathbb{Q}}$ is surely not even closed under $x \mapsto x^\tau$. In fact, $2^{\mathbb{Q}}$ is orthogonal to these power functions in the following sense: for $n \geq 2$ and for every $\mathbb{Q}$-linearly independent $p_1, ..., p_n \in \mathbb{Q}(\tau)$,

$$(2^{p_1})^{\mathbb{Q}} \cap (2^{p_2})^{\mathbb{Q}} \cdot ... \cdot (2^{p_n})^{\mathbb{Q}} = \{1\}.$$

We need to fix some notation. Let M be a model of the theory of $\tilde{\mathbb{R}}$. Let $m \in \mathbb{N}$ and let $a_1, ..., a_m \in M$ and $\vec{p_i} = (p_{i,1}, ..., p_{i,|\vec{p_i}|}) \in \mathbb{Q}(\tau)^{|\vec{p_i}|}$ for $i = 1, ..., m$. We write $a_i^{\vec{p_i}}$ for the tuple

$$(a_i^{p_{i,1}}, ..., a_i^{p_{i,n}}) \in M^{|\vec{p_i}|}.$$

We say that $a_1^{\vec{p_1}}, ..., a_m^{\vec{p_m}}$ are *multiplicatively dependent* if there are $\vec{n_i} \in \mathbb{Z}^{|\vec{p_i}|}$ for $i = 1, ..., m$ such that

$$\prod_{i=1}^{n} \prod_{j=1}^{|\vec{p_i}|} a_i^{n_{i,j} p_{i,j}} = 1.$$

The main difference between the theory of $\tilde{\mathbb{R}}$ and the theory of $\overline{\mathbb{R}}$ is that the theory of $\tilde{\mathbb{R}}$, though model complete, does not eliminate quantifiers. In order to handle the difficulties arising from this difference, we need to make use of the Schanuel condition. Let $m \in \mathbb{N}$ and for every $i = 1, ..., m$ let $\vec{p_i}$ be an element of $\mathbb{Q}(\tau)^{|\vec{p_i}|}$, whose coordinates are linearly independent over $\mathbb{Q}$. The *Schanuel condition* states that for every $a_1, ..., a_m \in \mathbb{R}^m_{>0}$, if

$$tr.deg_{\mathbb{Q}}(a_1^{\vec{p_1}}, ..., a_m^{\vec{p_m}}) < \sum_{i=1}^{m} |\vec{p_i}| - m,$$

then $a_1^{\vec{p_1}}, ..., a_m^{\vec{p_m}}$ are multiplicatively dependent (see Condition 3.1.2). We will use this condition and a uniform version of it (see Condition 3.1.4), in order to give a first order axiomatization of the structures investigated. It is known that these conditions actually hold for co-countably many $\tau \in \mathbb{R}$. In Chapter 3 Section 2 we will give a proof of this result which is nearly exactly Wilkie's proof in [W03].
For the following, we fix a multiplicative subgroup Γ of $\mathbb{R}$. Let $\mathfrak{L}^\tau$ be the language of $\tilde{\mathbb{R}}$ and let $\mathfrak{L}^\tau_\Gamma$ be the language $\mathfrak{L}^\tau$ plus one constant symbol $\dot{\gamma}$ for every $\gamma \in \Gamma$. Further let $\mathfrak{L}^\tau_\Gamma(G)$ be the language $\mathfrak{L}^\tau_\Gamma$ plus an extra unary predicate G. Note that $(\tilde{\mathbb{R}}, (\gamma \in \Gamma))$ is a $\mathfrak{L}^\tau_\Gamma$-structure and let T be its theory. We will always identify Γ with the set of the interpretations of all constants $\dot{\gamma}$, where $\gamma \in \Gamma$.

1.2.1 The theory of groups orthogonal to irrational power functions.

First consider dense multiplicative subgroups of $\mathbb{R}$ satisfying

(G1) every $g \in \Gamma$ is algebraic over $\mathbb{Q}$,

(G2) $|\Gamma : \Gamma^{[n]}| < \infty$ for $n \in \mathbb{N}_{>0}$,

(G3) for every $\mathbb{Q}$-linearly independent $(p_1, ..., p_n) \in \mathbb{Q}(\tau)^n$, the group $\Gamma^{[p_1]} \cdot \Gamma^{[p_2]} ... \cdot \Gamma^{[p_n]}$ has the Mann property,

(G4) for $n \geq 2$, for every $\mathbb{Q}$-linearly independent $(p_1, ..., p_n) \in \mathbb{Q}(\tau)^n$,

$$\Gamma^{[p_1]} \cap \Gamma^{[p_2]} \cdot ... \cdot \Gamma^{[p_n]} = \{1\},$$

where for every $p \in \mathbb{Q}(\tau)$, $\Gamma^{[p]}$ is the set $\{g^p | g \in \Gamma\}$. Note that (G1) implies that Γ is countable. Further, for every τ the group $2^{\mathbb{Q}}$ satisfies (G1)-(G4) (see Proposition 4.1.2 for a proof). We consider the class of all $\mathfrak{L}_\Gamma^\tau(G)$-structure (M, G) satisfying the following not obviously first-order axioms:

(A1) M is a model of T,

(A2) G is a dense multiplicative subgroup of M with pure subgroup Γ,

(A3) $|\Gamma : \Gamma^{[n]}| = |G : G^{[n]}|$, for all $n \in \mathbb{N}$,

(A4) all Mann solutions of $G^{[p_1]} \cdot G^{[p_2]} \cdot ... \cdot G^{[p_n]}$ are actually in the subgroup $\Gamma \cdot \Gamma^{[p_1]} \cdot ... \cdot \Gamma^{[p_n]}$, for all $(p_1, ..., p_n) \in \mathbb{Q}(\tau)^n$,

(A5) for $n \geq 2$, for every $\mathbb{Q}$-linearly independent $(p_1, ..., p_n) \in \mathbb{Q}(\tau)^n$,

$$G^{[p_1]} \cap G^{[p_2]} \cdot ... \cdot G^{[p_n]} = \{1\},$$

(A6) for all $\vec{p} \in \mathbb{Q}(\tau)^{|\vec{p}|}$, if $g_1, ..., g_m \in G$ and $y_1, ...y_n \in M$ satisfy

$$tr.deg_{\mathbb{Q}(g_1^{\vec{p}}, ..., g_m^{\vec{p}})}(y_1^{\vec{p}}, ..., y_n^{\vec{p}}) < n|\vec{p}| - n,$$

then $g_1^{\vec{p}}, ..., g_m^{\vec{p}}, y_1^{\vec{p}}, ..., y_n^{\vec{p}}$ are multiplicatively dependent,

(A7) for every $\mathfrak{L}_\Gamma^\tau$-0-definable function $f(x_1, ..., x_n)$, the set

$$\{a \in M | \forall g_1, ..., g_n \in G \ f(g_1, ..., g_n) \neq a\}$$

is dense in M.

We will show that this set of axioms is actually first-order under Schanuel condition 3.1.4 and we will call this theory $T \cup \mathrm{dmG}_\Gamma$. Our first main result is then

Theorem 1.2.2 *Assume Schanuel condition 3.1.4 for τ. Then $(\tilde{\mathbb{R}}, \Gamma) \models T \cup dmG_\Gamma$. Further $T \cup dmG_\Gamma$ is complete and every subset of $\mathbb{R}^m$ definable in $(\tilde{\mathbb{R}}, \Gamma)$ is a boolean combination of subsets of $\mathbb{R}^m$ defined in $(\tilde{\mathbb{R}}, \Gamma)$ by a formula of the form*

$$\exists y_1 ... \exists y_n \exists z_1 ... \exists z_l \; G(y_1) \wedge ... \wedge G(y_n) \wedge \varphi(x_1, ..., x_m, y_1, ..., y_n, z_1, ..., z_l),$$

where φ is a quantifier-free $\mathfrak{L}^\tau_\Gamma$-formula.

Remember that Condition 3.1.4 is known to be true for co-countably many $\tau \in \mathbb{R}$. Further in Chapter 6, we will use the work of Berenstein, Ealy and Günaydin from [BEG07] to show that

Theorem 1.2.3 *Assume Schanuel condition 3.1.4 for τ. Then $(\tilde{\mathbb{R}}, \Gamma)$ has o-minimal open core.*

In fact, this implies that Theorem 1.2.2 and Theorem 1.2.3 hold for a wider class of groups. Let $d \in \mathbb{N}$. Suppose Γ satisfies (G1)-(G4) and suppose τ is not algebraic of degree less or equal than d. Of course, the group $\Gamma \cdot \Gamma^{[\tau]} \cdot ... \cdot \Gamma^{[\tau^{d-1}]}$ is definable in $(\tilde{\mathbb{R}}, \Gamma)$. Since τ is not algebraic of degree less or equal than d, we have that $\gamma^{\tau^d} \notin \Gamma \cdot \Gamma^{[\tau]} \cdot ... \cdot \Gamma^{[\tau^{d-1}]}$ for every $\gamma \in \Gamma$. Hence the subgroup $\Gamma \cdot \Gamma^{[\tau]} \cdot ... \cdot \Gamma^{[\tau^{d-1}]}$ is not closed under all power functions and further Γ is definable in $(\tilde{\mathbb{R}}, \Gamma \cdot \Gamma^{[\tau]} \cdot ... \cdot \Gamma^{[\tau^{d-1}]})$ by the $\mathfrak{L}^\tau(G)$-formula

$$G(x) \wedge G(x^\tau) \wedge ... \wedge G(x^{\tau^{d-1}}).$$

This implies that $(\tilde{\mathbb{R}}, \Gamma)$ and $(\tilde{\mathbb{R}}, \Gamma \cdot \Gamma^{[\tau]} \cdot ... \cdot \Gamma^{[\tau^{d-1}]})$ have the same definable subsets. In particular, Theorem 1.2.2 and Theorem 1.2.3 also hold for this subgroup:

Theorem 1.2.4 *Assume Schanuel condition 3.1.4 for τ and τ is not algebraic of degree less or equal than d. Then $(\tilde{\mathbb{R}}, \Gamma \cdot \Gamma^{[\tau]} \cdot ... \cdot \Gamma^{[\tau^{d-1}]})$ has o-minimal open core and every subset of $\mathbb{R}^m$ definable in $(\tilde{\mathbb{R}}, \Gamma \cdot \Gamma^{[\tau]} \cdot ... \cdot \Gamma^{[\tau^{d-1}]})$ is a boolean combination of subsets of $\mathbb{R}^m$ defined in $(\tilde{\mathbb{R}}, \Gamma \cdot \Gamma^{[\tau]} \cdot ... \cdot \Gamma^{[\tau^{d-1}]})$ by a formula of the form*

$$\exists y_1 ... \exists y_n \exists z_1 ... \exists z_l \; G(y_1) \wedge ... \wedge G(y_n) \wedge \varphi(x_1, ..., x_m, y_1, ..., y_n, z_1, ..., z_l),$$

where φ is a quantifier-free $\mathfrak{L}^\tau_\Gamma$-formula.

Further note that for algebraic τ, the closure of Γ under all power functions is also definable in $(\tilde{\mathbb{R}}, \Gamma)$. Unfortunately, the above Theorem does not apply to the group $2^{\mathbb{Z}} \cdot 2^{\mathbb{Z}\tau}$, since $2^{\mathbb{Z}}$ is not a *dense* subgroup of $\mathbb{R}^\times$.

1.2.2 The theory of groups closed under all definable power functions.

We will also examine dense multiplicatve subgroups of $\mathbb{R}$ with the following property

(H1) there is $I \subseteq \overline{\mathbb{Q}}$ such that $\Gamma = \{g_1^{p_1} \cdot ... \cdot g_n^{p_n} \mid n \in \mathbb{N}, g_i \in I, p_i \in \mathbb{Q}(\tau)\}$, and

(H2) Γ has the Mann property.

We call groups satisfying (H1) *algebraically* $\mathbb{Q}(\tau)$*-generated*. Note that (H1) implies that Γ is divisible and countable. Under Schanuel condition 3.1.2 for τ, $2^{\mathbb{Q}(\tau)}$ satisfies (H1) and (H2) (see Theorem 3.4.5 for a proof). In this case, consider the class of all $\mathfrak{L}_\Gamma^\tau(G)$-structure (M, G) satisfying the following axioms:

(B1) M is a model of T.

(B2) G is a dense multiplicative subgroup of M with subgroup Γ and is closed under all power functions definable in $\tilde{\mathbb{R}}$,

(B3) all Mann solutions of G are actually in the subgroup Γ,

(B4) for all $\vec{p} \in \mathbb{Q}(\tau)^{|\vec{p}|}$, if $g_1, ..., g_m \in G$ and $y_1, ...y_n \in M$ satisfy

$$tr.deg_{\mathbb{Q}(g_1^{\vec{p}},...,g_m^{\vec{p}})}(y_1^{\vec{p}}, ..., y_n^{\vec{p}}) < n|\vec{p}| - n,$$

then $y_1, ..., y_n$ are multiplicatively dependent over G,

(B5) for every $\mathfrak{L}_\Gamma^\tau$-0-definable function $f(x_1, ..., x_n)$, the set

$$\{a \in M | \forall g_1, ..., g_n \in G \ f(g_1, ..., g_n) \neq a\}$$

is dense in M.

The case of groups closed under all definable power functions is more complicated, since not every group element is algebraic. Nevertheless, it will be shown that in two cases these axioms are first-order definable and we will call this theory $T \cup KmdG$ (see Chapter 5 Section 4). These two cases are the following:

(i) τ is algebraic and the Uniform Schanuel Condition 3.1.4 holds for τ,

(ii) τ is arbitrary, Schanuel Condition 3.1.2 holds for τ, the Conjecture on intersection with tori 5.3.3 holds and further the set I witnessing (H1) for Γ is finite.

Theorem 1.2.5 *Let $\tau \in \mathbb{R}$ either satisfy (i) or (ii). Then $(\tilde{\mathbb{R}}, \Gamma) \models T \cup KdmG$. Further $T \cup KdmG$ is complete and every subset of $\mathbb{R}^m$ definable in $(\tilde{\mathbb{R}}, \Gamma)$ is a boolean combination of subsets of $\mathbb{R}^m$ defined in $(\tilde{\mathbb{R}}, \Gamma)$ by a formula of the form*

$$\exists y_1 ... \exists y_n \exists z_1 ... \exists z_l \; G(y_1) \wedge ... \wedge G(y_n) \wedge \varphi(x_1, ..., x_m, y_1, ..., y_n, z_1, ..., z_l),$$

where φ is a quantifier-free $\mathfrak{L}_\Gamma^\tau$-formula.

Note that it is not known whether the Uniform Schanuel Condition 3.1.4 holds for any algebraic τ. Nor is it known whether the Conjecture on intersection with tori holds.
Again as in the case of the orthogonal groups, it will be shown in Chapter 6 that

Theorem 1.2.6 *Let $\tau \in \mathbb{R}$ satisfy either (i) or (ii). Then $(\tilde{\mathbb{R}}, \Gamma)$ has o-minimal open core.*

Further in Chapter 6 it will be shown that the case of groups closed under all definable power functions can not be reduced trivially to the case of groups orthogonal to the irrational power functions.

Theorem 1.2.7 *Let $\tau \in \mathbb{R}$ either satisfy (i) or (ii). Then $2^{\mathbb{Q}}$ is* not *definable in $(\tilde{\mathbb{R}}, 2^{\mathbb{Q}(\tau)})$.*

1.2.3 Irrational powers and $2^{\mathbb{Z}}$

Another open question is whether the structure consisting of $\tilde{\mathbb{R}}$ and the discrete group $2^{\mathbb{Z}}$ is tame in some sense. In Chapter 7 we will show that for this expansion Theorem 1.1.1 does not hold anymore.

Theorem 1.2.8 $(\tilde{\mathbb{R}}, 2^{\mathbb{Z}})$ *is not model complete.*

It is not known whether $(\tilde{\mathbb{R}}, 2^{\mathbb{Z}})$ is near model complete or whether it defines the integers. There is however a close connection between this structure and the Diophantine approximations of τ. In Chapter 7 Section 3 it will be shown that the definable sets of $(\tilde{\mathbb{R}}, 2^{\mathbb{Z}})$ depend on the Diophantine approximation of τ and that some very special quantifiers can be eliminated for almost all τ. It is not clear whether these results can be used to prove a more general quantifier elimination statement. Hence the question about near model completeness and definability of $\mathbb{Z}$ remains open.

Finally note that if τ is not algebraic of order 2, then the subgroup $2^{\mathbb{Z}}2^{\mathbb{Z}\tau}$ is dense and $2^{\mathbb{Z}}$ is definable in $(\tilde{\mathbb{R}}, 2^{\mathbb{Z}}2^{\mathbb{Z}\tau})$ by

$$\{x \in 2^{\mathbb{Z}}2^{\mathbb{Z}\tau} \mid x^{\tau} \in 2^{\mathbb{Z}}2^{\mathbb{Z}\tau}\}.$$

Hence showing any tameness result for $(\tilde{\mathbb{R}}, 2^{\mathbb{Z}})$ is equivalent to showing a tameness result for $(\tilde{\mathbb{R}}, 2^{\mathbb{Z}}2^{\mathbb{Z}\tau})$.

1.3 Outline of the main argument

We will now outline the main argument and the structure of this thesis. The crucial part of the proof of Theorem 1.1.3 is the following lemma:

Lemma 1.3.1 *[[vdDG06] Lemma 5.12.] Let F be a real closed field and G a multiplicative subgroup. Let Γ be a subgroup of G such that for all $n \geq 1$ and $a_1, ..., a_n \in \mathbb{Q}$ the equation $a_1x_1 + ... + a_nx_n = 1$ has the same non-degenerate solutions in Γ as in G. If $g \in G$ is algebraic over $\mathbb{Q}(\Gamma)$ of degree d, then $g^d \in \Gamma$.*

The main difficulty in proving Theorem 1.2.2 is establishing the following generalization of Lemma 1.3.1. Consider a multiplicative subgroup Γ of $\mathbb{R}^{\times}$ satisfying (G1)-(G4).

Theorem 1.3.2 *Let $(M, G) \models T \cup dmG_{\Gamma}$ and H be a pure subgroup of G, which contains all interpretations of the constants $\dot{\gamma}$, where $\gamma \in \Gamma$. Then*

$$\boldsymbol{cl}_T(H) \cap G = H,$$

where $\boldsymbol{cl}_T(\cdot)$ denotes the definable closure operation in T.

The difference between the proof of Lemma 1.3.1 and Theorem 1.3.2 comes from the fact that T does not eliminate quantifiers. We now give a sketch of the argument of the proof of Theorem 1.3.2. Suppose there is $g \in \Gamma$ such that $g \in \mathbf{cl}_T(h_1, ..., h_m)$, where $h_1, ..., h_m \in \Gamma$. We can assume by model completeness that there are $y_1, ..., y_n \in M$ and $\vec{p} \in \mathbb{Q}(\tau)^l$ and polynomials $q_1, ..., q_{n+1}$ with coefficients in $\mathbb{Q}$ such that

$$\begin{aligned} q_1(h_1^{\vec{p}}, ..., h_m^{\vec{p}}, g^{\vec{p}}, y_1^{\vec{p}}, ..., y_n^{\vec{p}}) &= 0 \\ \vdots \qquad\qquad & \vdots \\ q_{n+1}(h_1^{\vec{p}}, ..., h_m^{\vec{p}}, g^{\vec{p}}, y_1^{\vec{p}}, ..., y_n^{\vec{p}}) &= 0. \end{aligned}$$

and hence

$$tr.deg_{\mathbb{Q}(h_1^{\vec{p}},...,h_m^{\vec{p}},g^{\vec{p}})}(y_1^{\vec{p}},...,y_n^{\vec{p}}) < n|\vec{p}| - n. \qquad (1.3.1)$$

See Chapter 2 Section 3 for details. Now assume $n > 0$ and minimal with the above property. We can further assume that

$$h_1^{\vec{p}},...,h_m^{\vec{p}},g^{\vec{p}},y_1^{\vec{p}},...,y_n^{\vec{p}} \text{ are multiplicatively independent.} \qquad (1.3.2)$$

See Chapter 2 Section 4 for details. Since $h_1,...,h_n,g \in \Gamma$, we know that $h_1,...,h_m,g$ are algebraic. Hence

$$tr.deg_{\mathbb{Q}}(h_1^{\vec{p}},...,h_m^{\vec{p}},g^{\vec{p}},y_1^{\vec{p}},...,y_n^{\vec{p}}) < (n+m+1)|\vec{p}| - (n+m+1). \qquad (1.3.3)$$

By the Schanuel Condition 3.1.2, this contradicts (1.3.2). Hence we can assume that $n = 0$ and hence that there is a polynomial q with coefficients in $\mathbb{Q}$ such that

$$q(h_1^{\vec{p}},...,h_m^{\vec{p}},g^{\vec{p}}) = 0.$$

In this case the methods of the proof of Lemma 1.3.1 can be used.
In the above argument, we have assumed $h_1,...,h_n,g$ to be in Γ rather than in G. But using the Uniform Schanuel Condition 3.1.4, we can show that axiom (A5) holds in $(\tilde{\mathbb{R}},\Gamma)$ and is first-order expressible (see Chapter 4 Section 2). This allows us to conclude multiplicative dependency from inequality (1.3.1), even if $h_1,...,h_n,g$ are not algebraic themselves and hence inequality (1.3.3) does not necessarily hold (see Chapter 4.4 for details). In the rest of Chapter 4, it will be shown that Theorem 1.2.2 follows from Theorem 1.3.2 as Theorem 1.1.3 follows from Lemma 1.3.1.
A result similar to Theorem 1.3.2 also holds for subgroups Γ which satisfy (H1)-(H2) instead of (G1)-(G4). But the proof is slightly different, because not every element of Γ is algebraic. In fact, in order to show that axiom (B4) is first-order expressible, we need to assume that either τ is algebraic or the Conjecture on intersection with tori holds. The details can be found in Chapter 5.
In Chapter 6 we will show that Theorem 1.3.2 and Theorem 1.2.2 imply that under certain sufficient assumptions, $(\tilde{\mathbb{R}},\Gamma)$ has o-minimal open core. As already noted, in [BEG07] Berenstein, Ealy and Günaydin used Lemma 1.3.1 and Theorem 1.1.3 to show that the real field together with a dense multiplicative subgroup satisfying the Mann property has o-minimal open core. We will see that their proof can easily be adjusted to the situation of $(\tilde{\mathbb{R}},\Gamma)$.
Finally Chapter 7 is independent of the rest of the thesis. It will consider the previously mentioned results on $(\tilde{\mathbb{R}},2^{\mathbb{Z}})$ and $(\tilde{\mathbb{R}},2^{\mathbb{Z}}2^{\mathbb{Z}\tau})$.

Chapter 2

O-minimality

In this chapter we will review the basic notions used in the text. In particular, polynomially bounded o-minimal structures will be defined and further examples will be given. In Section 2 we will review Miller's results from [M94] on adding power functions to o-minimal structures. Finally in Section 3, the structure consisting of the real field and one irrational power function will be the focus of the discussion. Jones and Wilkie's results from [JW08] will be used to give a characterization of definable sets in this structure.

2.1 Basic definitions

In this first section, we will recall the notion of o-minimality and state some basic results.

Definition 2.1.1 *A structure $\mathcal{R} = (R, <, ...)$ is called* o-minimal*, if $<$ is a dense linear order without endpoints on R and every definable set $S \subseteq R$ is a union of finitely many intervals and points.*

We will now state several classical result for o-minimal theories, which will be used in the rest of this text. For proofs, see [vdD98]. Let now $\mathcal{R} = (R, <, +, -, \cdot, 0, 1, ...)$ be an o-minimal extension of an ordered field. Let $a, b \in R \cup \{-\infty, \infty\}$.

Theorem 2.1.2 *(Monotonicity Theorem) Let $f : (a, b) \to R$ be definable. Then there are points $a_1 < ... < a_k$ in (a, b) such that on each subinterval (a_j, a_{j+1}) the function f is continuous and either constant or strictly monotonic, where $a_0 = a$ and $a_{k+1} = b$.*

Theorem 2.1.3 *Let $f : (a, b) \to R$ be definable. Then there are $a_1, ..., a_n \in (a, b)$ such that f is differentiable at all points $x \in (a, b) \backslash \{a_1, ..., a_n\}$.*

A consequence of these two theorems is that every definable function $f : R \to R$ is ultimately continuous and differentiable. Further in $\mathcal{R}$ the Theorem of Rolle holds and so does its important consequence, the Mean Value Theorem. Let $a, b \in R$.

Theorem 2.1.4 *(Mean Value Theorem) Let $f : [a, b] \to R$ be definable, continuous and differentiable at all points in (a, b). Then there is a $c \in (a, b)$ such that*

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

This implies that every o-minimal expansion of an ordered field is in fact an expansion of a real closed field.

The Monotonicity Theorem also generalizes to higher dimension. Therefore we need the following definition of a cell.

Definition 2.1.5 *A cell is a definable subset of R^n obtained by induction as follows*

(i) $\{r\} \subset R$ *is a cell for every* $r \in R$*,*

(ii) the interval $(r_1, r_2) \subset R$ *is a cell for every* $r_1, r_2 \in R \cup \{\pm\infty\}$*,*

(iii) if $C \subseteq R^n$ *is a cell and* $f : C \to R$ *is a continuous definable function, then the graph of* f *is a cell as well,*

(iv) if $C \subseteq R^n$ *is a cell and* $f, g : C \to R$ *are either continuous definable function or identically* $-\infty$ *or* $+\infty$*, then*

$$\{(\vec{a}, b) \in R^{n+1} \mid \vec{a} \in C \wedge f(\vec{a}) < b < g(\vec{a})\}$$

is a cell.

Theorem 2.1.6 *For every definable set* X *in* $\mathcal{R}$*, there are finitely many cells* $C_1, ..., C_n$ *such that*

$$X = C_1 \cup ... \cup C_n.$$

Further note that every o-minimal group $\mathcal{S} = (S, <, +, 0)$ has only the trivial definable subgroups. Thus in the case of an o-minimal expansion of an ordered field $\mathcal{R}$, we get that

Proposition 2.1.7 $(R, +, 0)$ *and* $(R^{>0}, \cdot, 1)$ *have only trivial definable subgroups.*

Another nice property of an o-minimal expansion of an ordered field $\mathcal{R}$ is the existence of definable choice functions.

Theorem 2.1.8 *(Definable Choice) If* $A \subseteq R^{m+n}$ *is definable, and* $\pi : R^{m+n} \to R^m$ *is the projection to the first* m *coordinates, then there is a definable function* $f : \pi(A) \to R^n$ *such that the graph of* f *is a subset of* A*.*

As a corollary, we get that for any two models $\mathcal{R}$ and $\mathcal{S}$ of a complete o-minimal theory T extending RCF with $\mathcal{R} \prec \mathcal{S}$, and for any $\vec{a} \in S^n$, the structure $\mathcal{R}\langle\vec{a}\rangle := \{f(\vec{a}) | f \text{ is a definable function in } \mathcal{R}\}$ is a model of T and $\mathcal{R} \preceq \mathcal{R}\langle\vec{a}\rangle$. Further note that this implies that the set of 0-definable elements $\mathcal{P}$ in T is the prime model of T. Given $\vec{a}$ in any model $\mathcal{S}$ of T, we write $\mathbf{cl}_T(\vec{a})$ for $\mathcal{P}\langle\vec{a}\rangle$. We will call $\mathbf{cl}_T(\vec{a})$ *the definable closure of* $\vec{a}$ *under* T. The Monotonicity Theorem implies that $\mathbf{cl}_T$ has the Steinitz Exchange Property and hence that

Theorem 2.1.9 $\boldsymbol{cl}_T$ *is a pregeometry.*

2.1.1 Power-boundedness

Let $\mathcal{R} = (R, <, +, -, \cdot, 0, 1, ...)$ be an o-minimal expansion of an ordered field.

Definition 2.1.10 *A* power function *of* $\mathcal{R}$ *is a definable endomorphism of the multiplicative group* $(R^{>0}, \cdot, 1)$.

In fact, since $\mathcal{R}$ is real closed, for every $q \in \mathbb{Q}$ the function $x \mapsto x^q$ is a power function. Of course, the derivative of this function at 1 is q. In general, every power function f is differentiable. To see this note that

$$\frac{f(x+h) - f(x)}{h} = \frac{f(x)}{x} \cdot \frac{f(1 + \frac{h}{x}) - 1}{\frac{h}{x}}.$$

Since f is ultimately differentiable, it is differentiable at 1, and hence everywhere on $R^{>0}$ with $f'(x) = \frac{f(x)}{x} f'(1)$. Suppose now g is a second power function. By the standard rules of derivations, we have

$$(fg)'(1) = f'(1) + g'(1). \tag{2.1.1}$$

Note that $\frac{1}{g}$ and $\frac{f}{g}$ are also power functions. By equation (2.1.1) we get $(\frac{1}{g})'(1) = -g'(1)$. Further this gives us $(\frac{f}{g})'(1) = f'(1) - g'(1)$. So if $f'(1) = g'(1)$, then $(f/g)'(1) = 0$. Hence $(f/g)'(x) = 0$ for all $x \in R^{>0}$ and thus $f = g$. So every power function is uniquely determined by the value of its derivative at 1.

Definition 2.1.11 $K := \{f'(1) \mid f \text{ } \textit{is a power function}\}$ *is called the* field of exponents *of* $\mathcal{R}$.

We want to note some facts about the field of exponents. Firstly, K is a field. Equation (2.1.1) shows that K is closed under addition and substraction. Further 0 is the value of derivative of the constant power function $x \mapsto 1$ at 1 and 1 itself is

the value of the derivative of the power function $x \mapsto x$. So it is only left to show that K is closed under multiplication and taking the multiplicative inverse. Let f, g be two power functions, then $g \circ f$ is a power functions. By the chain rule, we get $(g \circ f)'(1) = f'(1)g'(1)$. Thus K is closed under multiplication. Now assume that f is not equal to $x \mapsto 1$. Since both the kernel and the image of f are definable subgroups of $(R^{>0}, \cdot, 1)$, the image of f is $R^{>0}$ and its kernel is $\{1\}$ by Proposition 2.1.7. Hence f is bijective. Let g be its inverse. Since g is a again a power function and $g'(1) = \frac{1}{f'(1)}$, we have that K is closed under taking the multiplicative inverse and hence K is a field.
Further note that $\mathbb{Q} \subseteq K$, since $x \mapsto x^q$ is a power function for all $q \in \mathbb{Q}$. In this spirit, for every $r \in K$ we write x^r for the power function f with $f'(1) = r$.

Definition 2.1.12 *We say $\mathcal{R}$ is* power-bounded*, if for all definable $f : R \to R$, there is a $r \in K$ such that for every big enough $x \in R$, $|f(x)| \leq x^r$.*

Definition 2.1.13 *We say $\mathcal{R}$ is* polynomially-bounded*, if for all definable $f : R \to R$, there is a $n \in \mathbb{N}$ such that for every big enough $x \in R$, $|f(x)| \leq x^n$.*

Note that for a power-bounded structure $\mathcal{R}$, $\mathcal{R}$ is polynomially bounded iff K is archimedean. Miller proved in [M96] that $\mathcal{R}$ is either power-bounded or an exponential function is 0-definable in $\mathcal{R}$. This implies that for every complete o-minimal theory T extending RCF, all models of T are power-bounded iff one model of T is power-bounded. We call such a theory T *power-bounded*. Further Miller showed that for power-bounded $\mathcal{R}$, every element $r \in K$ and its power function x^r are 0-definable and that for every theory T, every model of T has up to isomorphism the same field of exponents.
In the case of a power-bounded structure $\mathcal{R}$, for every definable function f there is a $c \in R$ and an $\alpha \in K$ such that $\lim_{x\to\infty}(f(x)/(cx^\alpha)) = 1$. Miller even proved that

Theorem 2.1.14 *[[M96] Theorem 4.2] Let $f : A \times R \to R$ be definable, $A \subseteq R^m$, such that for all $a \in A$ the function $x \mapsto f(a, x)$ is ultimately nonzero. Then there exists $\alpha_1, ..., \alpha_n \in K$ and a definable function $c : A \to R^\times$ such that for all $a \in A$*

$$\lim_{x\to\infty}(f(a, x)/(c(a)x^{\alpha_i})) = 1,$$

for some $i \in \{1, ..., n\}$.

2.2 Examples

In this section, we will give some examples of the notions defined in the previous section. The easiest example of an o-minimal structure is $(R, <)$, where $<$ is a dense linear order. By the Tarski-Seidenberg Theorem, ie. the fact that the theory of real closed fields has quantifier elimination, every real closed field, especially $\overline{\mathbb{R}} := (\mathbb{R}, <, +, -, \cdot, 0, 1)$, is o-minimal. In [W96] Wilkie showed that also the field of real numbers together with the exponential function is o-minimal.

Another important example of an o-minimal structure mentioned in the introduction is the field of real numbers $\mathbb{R}$ together with all restricted analytic function. This structure is usually denoted by $\mathbb{R}_{an}$ and defined as follows. Let $I = [-1, 1]$ and let $\mathcal{J}$ be the set of analytic functions defined on a neighborhood of I^m. Then for every $f \in \mathcal{J}$ define

$$\tilde{f}(x) := \begin{cases} f(x), & x \in I^m; \\ 0, & \text{otherwise.} \end{cases}$$

Now let $\mathbb{R}_{an}$ be the structure consisting of the real field and all functions $\tilde{f}$ for $f \in \mathcal{J}$. In [vdD86] van den Dries showed that $\mathbb{R}_{an}$ is o-minimal and polynomially bounded with field of exponents $\mathbb{Q}$. Note in particular that polynomially boundedness implies that the exponential function is *not* definable in $\mathbb{R}_{an}$. But in fact its restriction to any subinterval is definable, as is the restriction of the logarithm function. As noted in the first chapter, the function $\log_2|_{(2,4)}$ is definable in $\mathbb{R}_{an}$.

The next structure we want to consider is the structure of the real field together with an irrational power function. Therefore let $\tau \in \mathbb{R} - \mathbb{Q}$, and let $f_\tau : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) := \begin{cases} x^\tau, & x > 0; \\ 0, & \text{otherwise.} \end{cases}$$

Now let $\tilde{\mathbb{R}_\tau}$ be the structure $(\overline{\mathbb{R}}, f_\tau)$. In [M94] Miller showed that the structure $\tilde{\mathbb{R}_\tau}$ is o-minimal, model complete and polynomially-bounded with field of exponents $\mathbb{Q}(\tau)$. In the following, we will sometimes write $\tilde{\mathbb{R}}$ instead of $\tilde{\mathbb{R}_\tau}$ if it is clear which τ is meant.

2.3 Definability in $\tilde{\mathbb{R}_\tau}$

In this section we will take a closer look at the structure $\tilde{\mathbb{R}_\tau}$. Let T be the theory of $\tilde{\mathbb{R}_\tau}$. We will analyze when an element b is definable over other elements $a_1, .., a_n$. It will be shown how multiplicative dependencies can be used to simplify the way b is definable by $a_1, ..., a_n$.

2.3.1 Regular solutions

First we need to fix some notation. Given a polynomial $q \in \mathbb{Q}[\vec{X}, \vec{Y}]$, where $\vec{X}, \vec{Y}$ are not necessarily of the same length, and a vector $\vec{b}$ of elements in some field extension of $\mathbb{Q}$, we will write $q_{\vec{b}}$ for the polynomial

$$q(\vec{b}, -) \in \mathbb{Q}(\vec{b})[\vec{Y}].$$

Further, let M be a model of T and $f_1, ..., f_p : M^n \to M$ be differentiable with $p \leq n$. The *Jacobian matrix* of $f_1, ..., f_p$ is defined as the matrix

$$J_n(f_1, ..., f_p) := \begin{pmatrix} \nabla f_1 \\ \vdots \\ \nabla f_p, \end{pmatrix}$$

where $\nabla f : M^n \to M^n$ is the function defined by

$$\vec{a} \mapsto \langle \frac{\partial f}{\partial x_1}(\vec{a}), ..., \frac{\partial f}{\partial x_n}(\vec{a}) \rangle.$$

Finally let $\mathfrak{Q}_{n,f_1,...,f_p} : M^n \to M$ be the function which maps $\vec{a} \in M^n$ to the sum of squares of all determinants of $p \times p$ submatrices of $J_n(f_1, ..., f_p)(\vec{a})$.
As already mentioned in the introduction, for every element $a \in M$ and every tuple $\vec{p} = (p_1, ..., p_n) \in \mathbb{Q}(\tau)^n$, we write $a^{\vec{p}}$ for the tuple

$$(a^{p_1}, ..., a^{p_n}) \in M^n.$$

Now we turn to a result on definability in $\tilde{\mathbb{R}}$ which follows from work of Jones and Wilkie in [JW08]. The following proposition is an instance of their Theorem 4.2 with $\mathcal{F}$ being $\{x \mapsto x^\tau\}$.

Proposition 2.3.1 *Let $M \models T$ and $a_1, ..., a_m \in M$. If $b \in \boldsymbol{cl}_T(a_1, ..., a_m)$, there is $n \in \mathbb{N}$, $\vec{p} \in \mathbb{Q}(\tau)^{|\vec{p}|}$ and for every $i \in \{0, ..., n\}$, there is*

- $c_i \in M$

and for every $i \in \{1, ..., n+1\}$ there is

- $q_i \in \mathbb{Q}[X_1, ..., X_d]$*, where* $d := (m+n+1)|\vec{p}|$*,* *such that* $c_0 = b$ *and*

$$q_1(a_1^{\vec{p}}, ..., a_m^{\vec{p}}, c_0^{\vec{p}}, ..., c_n^{\vec{p}}) = 0,$$
$$\vdots \qquad \vdots$$
$$q_{n+1}(a_1^{\vec{p}}, ..., a_m^{\vec{p}}, c_0^{\vec{p}}, ..., c_n^{\vec{p}}) = 0,$$

and

$$\mathfrak{Q}_{n+1,Q_1,...,Q_{n+1}}(c_0, ..., c_n) \neq 0, \tag{2.3.1}$$

where $Q_i : M^{n+1} \to M$ *is the function defined by*

$$Q_i(y_0, y_1, ..., y_n) := q_i(a_1^{\vec{p}}, ..., a_m^{\vec{p}}, y_0^{\vec{p}}, y_1^{\vec{p}}, ..., y_n^{\vec{p}})$$

Definition 2.3.2 *With the notation of Proposition 2.3.1, we say that the* $c_1, ..., c_n$ define b over $a_1, ..., a_m$ via $(\vec{p}, q_1, ..., q_{n+1})$.

Corollary 2.3.3 *With the notation of Proposition 2.3.1,*

$$\text{if } \mathfrak{Q}_{n+1,Q_1,...,Q_{n+1}}(c_0, ..., c_n) \neq 0, \text{ then } \mathfrak{Q}_{\eta,q_{1,\vec{a}},...,q_{n+1,\vec{a}}}(c_0^{\vec{p}}, ..., c_n^{\vec{p}}) \neq 0, \tag{2.3.2}$$

where $\eta := (n+1)|\vec{p}|$ *and* $\vec{a} := (a_1^{\vec{p}}, ..., a_m^{\vec{p}})$

Proof: By the chain rule, we know that for $k = 1, ..., n+1$ and $i = 0, ..., n$,

$$\frac{\partial Q_k}{\partial y_i}(c_0, ..., c_n) = \sum_{j=1}^{|\vec{p}|} \frac{\partial q_{k,\vec{a}}}{\partial x_{j+i|\vec{p}|}}(c_0^{\vec{p}}, ..., c_n^{\vec{p}}) p_j c_i^{p_j - 1}.$$

Hence we get by elementary manipulations of columns and of rows of $J_{\eta,q_{1,\vec{a}},...,q_{n+1,\vec{a}}}(c_0^{\vec{p}}, ..., c_n^{\vec{p}})$, a matrix with a $(n+1) \times (n+1)$-submatrix which is equal to $J_{n+1,Q_1,...,Q_{n+1}}(c_0, ..., c_n)$. Since the determinant of the latter one is non-zero, this is also true for some $(n+1) \times (n+1)$-submatrix of $J_{\eta,q_{1,\vec{a}},...,q_{n+1,\vec{a}}}(c_0^{\vec{p}}, ..., c_n^{\vec{p}})$. By definitions, this implies that the condition on the right hand side of (2.3.2) holds.

□

Hence Proposition 2.3.1 and Corollary 2.3.3 imply

Corollary 2.3.4 *Let* $M \models T$ *and* $a_1, ..., a_m, b, c_1, ..., c_n \in M$. *If* $c_1, ..., c_n$ *define* b *over* $a_1, ..., a_m$, *then there is a* $\mathbb{Q}$*-linearly independent tuple* $\vec{p} \in \mathbb{Q}(\tau)^{|\vec{p}|}$ *such that*

$$tr.deg_{\mathbb{Q}(a_1^{\vec{p}}, ..., a_m^{\vec{p}})}(b^{\vec{p}}, c_1^{\vec{p}}, ..., c_n^{\vec{p}}) \leq (n+1)|\vec{p}| - (n+1).$$

2.3.2 Definability and multiplicative dependency

In fact, in Proposition 2.3.1 and so in Definition 2.3.2 we could assume that 1 occurs in the vector $\vec{p}$. Further by increasing n, we could assume that the variables representing $a_i^{p_i}$, where $p_i \neq 1$, do not appear in any of the polynomials $q_1, ...q_{n+1}$. If one would add this assumptions in Definition 2.3.2, the following Theorem 2.3.5 would not hold.

Theorem 2.3.5 *With notation of Proposition 2.3.1, let* $a_1, ..., a_m, c_0, ..., c_n \in M$ *and suppose that* $c_1, ..., c_n$ *define* b *over* $a_1, ..., a_m$. *If there are* $l_i, k_i \in \mathbb{Q}(\tau)$ *such that*

$$c_n = \prod_{i=1}^{m} a_i^{l_i} \cdot \prod_{i=0}^{n-1} c_i^{k_i}, \tag{2.3.3}$$

then $c_1, ..., c_{n-1}$ *define* b *over* $a_1, ..., a_m$.

Proof: We can assume that q_{n+1} witnesses the multiplicative dependency given by equation (2.3.3) and that 1, $k_0, ..., k_{n-1}$ and $l_1, ..., l_m$ occur in the vector $\vec{p}$. Define $V : M^n \to M$ by

$$V(y_0, ..., y_{n-1}) := \prod_{i=1}^{m} a_i^{l_i} \cdot \prod_{i=0}^{n-1} y_i^{k_i}.$$

Since q_{n+1} witnesses equation (2.3.3), we have

$$Q_{n+1}(y_0, ..., y_n) = y_n - V(y_0, ..., y_{n-1}).$$

This implies that $V(c_0, ..., c_{n-1}) = c_n$ and for $i = 1, ..., n-1$

$$\frac{\partial V}{\partial y_i}(c_0, ..., c_{n-1}) = -\frac{\partial Q_{n+1}}{\partial y_i}(c_0, ..., c_n). \tag{2.3.4}$$

Let $t_1, ..., t_l \in \mathbb{Q}(\tau)$ be such that $p_1, ..., p_{|\vec{p}|}, t_1, ..., t_l$ is a basis of the $\mathbb{Q}$-vector space generated by

$$\{p_1, ..., p_{|\vec{p}|}\} \cup \{p_j \cdot l_i | j = 1, ..., |\vec{p}|, i = 1, ..., m\} \cup \{p_j \cdot k_i | j = 1, ..., |\vec{p}|, i = 0, ..., n-1\}.$$

Set

$$\vec{p'} := (p_1, ..., p_{|\vec{p}|}, t_1, ..., t_l).$$

Hence for every $v = 1, ..., |\vec{p}|$, there exist $s_{i,j}, u_{i,j} \in \mathbb{Z}$ such that

$$c_n^{p_v} = \prod_{i=1}^{m} \prod_{j=1}^{|\vec{p'}|} a_i^{p'_j s_{i,j}} \cdot \prod_{i=0}^{n-1} \prod_{j=1}^{|\vec{p'}|} c_i^{p'_j u_{i,j}}. \tag{2.3.5}$$

We now replace the variables representing $c_n^{\vec{p}}$ in $q_1, ..., q_n$ according to equation (2.3.5). Let $r_1, ..., r_n \in \mathbb{Q}(X_1, ..., X_{(n+m)|\vec{p'}|})$ be the rational function obtained in this way. By the construction of r_i, there are $\alpha_{i,j} \in \mathbb{N}$ such that

$$w_i := r_i \prod_{j=1}^{(n+m)|\vec{p'}|} X_j^{\alpha_{i,j}} \in \mathbb{Q}[X_1, ..., X_{(n+m)|\vec{p'}|}] \tag{2.3.6}$$

Further for any $i = 1, ..., n$, define for $j = 1, ..., m$

$$\beta_{i,j} := \sum_{k=1}^{\vec{p'}} \alpha_{i,(j-1)\cdot|\vec{p}|+k} \cdot p'_k \in \mathbb{Q}(\tau)$$

and for $j = 0, ..., n-1$

$$\gamma_{i,j} := \sum_{k=1}^{\vec{p'}} \alpha_{i,(m+j)\cdot|\vec{p}|+k} \cdot p'_k \in \mathbb{Q}(\tau)$$

Let $W_i : M^{n+1} \to M$ be the function defined by

$$W_i(y_0, y_1, ..., y_{n-1}) := w_i(a_1^{\vec{p'}}, ..., a_m^{\vec{p'}}, y_0^{\vec{p'}}, ..., y_{n-1}^{\vec{p'}})$$

In order to show that $c_1, ..., c_{n-1}$ defines b over $a_1, ..., a_m$ via $(\vec{p'}, w_1, ..., w_n)$, it is only left to show that the matrix $J_n(W_1, ..., W_n)(\vec{c})$ has non-zero determinant. Therefore note that by equation (2.3.6) and the definition of r_i

$$W_i(y_0, y_1, ..., y_{n-1}) = \prod_{j=1}^{m} a_j^{\beta_{i,j}} \prod_{j=0}^{n-1} y_j^{\gamma_{i,j}} Q_i(y_0, y_1, ...y_{n-1}, V(y_0, y_1, ...y_{n-1})).$$

Set $\vec{c} := (c_0, ..., c_{n-1})$. Since $Q_i(\vec{c}, V(\vec{c})) = 0$, we get by equation (2.3.4) that

$$\begin{aligned}\frac{\partial W_i}{\partial y_j}(\vec{c}) &= \prod_{k=1}^{m} a_k^{\beta_{i,k}} \prod_{k=0}^{n-1} c_k^{\gamma_{i,k}} \left[\frac{\partial Q_i}{\partial y_j}(\vec{c}, V(\vec{c})) + \frac{\partial Q_i}{\partial y_n}(\vec{c}, V(\vec{c})) \cdot \frac{\partial V}{\partial y_j}(\vec{c})\right] \\ &= \prod_{k=1}^{m} a_k^{\beta_{i,k}} \prod_{k=0}^{n-1} c_k^{\gamma_{i,k}} \left[\frac{\partial Q_i}{\partial y_j}(\vec{c}, c_n) - \frac{\partial Q_i}{\partial y_n}(\vec{c}, c_n) \cdot \frac{\partial Q_{n+1}}{\partial y_j}(\vec{c}, c_n)\right].\end{aligned}$$

Note that $\frac{\partial Q_{n+1}}{\partial y_n}(\vec{c}, c_n) = 1$. Now perform the following operations on the matrix $J_{n+1}(Q_1, ..., Q_{n+1})(\vec{c}, c_n)$:

(i) For $i = 1, ..., n$ multiply the i-th row by $\prod_{k=1}^{m} a_k^{\beta_{i,k}} \prod_{k=0}^{n-1} c_k^{\gamma_{i,k}} \neq 0$,

(ii) For every $j = 0, ..., n-1$, add to the $j+1$-th column the $n+1$-th column multiplied by $-\frac{\partial Q_{n+1}}{\partial y_j}(\vec{c}, c_n)$.

By the above, this operations yield the following matrix

$$\begin{pmatrix} & & & * \\ & J_n(W_1,...,W_n)(\vec{c}) & & \vdots \\ & & & * \\ 0 & \cdots & 0 & 1 \end{pmatrix} \tag{2.3.7}$$

Since the matrix $J_{n+1}(Q_1,...,Q_{n+1})(\vec{c},c_n)$ has non-zero determinant, by (2.3.7) the matrix $J_n(W_1,...,W_n)(\vec{c})$ has also non-zero determinant.

□

Chapter 3

Schanuel conditions and the Mann property

In this chapter we consider the two main tools needed in the proofs of the results in the subsequent chapters. We will first consider certain uniform Schanuel conditions for a real number τ and show that these conditions hold for co-countably many real numbers. This will be achieved by slightly modifying the proof given by Wilkie in [W03]. Further the Mann property will be introduced and we will prove that for certain groups the Mann property is implied by the Schanuel condition. Finally, in Section 5, results from [RZ60] on regularly dense groups will be reviewed.

3.1 Schanuel condition

First consider the following Schanuel condition for $x \mapsto x^\tau$:

Condition 3.1.1 *Let $m \in \mathbb{N}$ and, for $i = 1, ..., m$, let $a_i \in \mathbb{R}$. If*

$$tr.deg_{\mathbb{Q}}(a_1, a_1^\tau, a_2, a_2^\tau, ..., a_m, a_m^\tau) < m$$

then $a_1, a_1^\tau, a_2, a_2^\tau, ..., a_m, a_m^\tau$ are multiplicatively dependent.

Wilkie showed in [W03] that Condition 3.1.1 holds at least for co-countably many $\tau \in \mathbb{R}$. For this thesis we have to consider more general Schanuel conditions. We will use the following convention: Let $m \in \mathbb{N}$, let $a_i \in \mathbb{R}$ and $\vec{p_i} \in \mathbb{Q}(\tau)^{|\vec{p_i}|}$ for $i = 1, ..., m$. We say that $a_1^{\vec{p_1}}, ..., a_m^{\vec{p_m}}$ are *multiplicatively dependent* if there are $\vec{n_i} \in \mathbb{Z}^{|\vec{p_i}|}$ for $i = 1, ..., m$ such that not all $n_{i,j}$'s are 0 and

$$\prod_{i=1}^{m} \prod_{j=1}^{|\vec{p_i}|} a_i^{n_{i,j} p_{i,j}} = 1.$$

Condition 3.1.2 *Let $m \in \mathbb{N}$ and for $i = 1, ..., m$ let $\vec{p_i} \in \mathbb{Q}(\tau)^{|\vec{p_i}|}$, whose coordinates are $\mathbb{Q}$-linearly independent. Then for every $(a_1, ..., a_m) \in \mathbb{R}^m$, if*

$$tr.deg_{\mathbb{Q}}(a_1^{\vec{p_1}}, ..., a_m^{\vec{p_m}}) < \sum_{i=1}^{m} |\vec{p_i}| - m,$$

then $a_1^{\vec{p_1}}, ..., a_m^{\vec{p_m}}$ are multiplicatively dependent.

One directly sees, that for $\vec{p_i} = (1, \tau)$, Condition 3.1.2 implies Condition 3.1.1. Now consider the following uniform version of the Schanuel condition.

Condition 3.1.3 *Let $m \in \mathbb{N}$, and for $i = 1, ..., m$ let $\vec{p_i} \in \mathbb{Q}(\tau)^{|\vec{p_i}|}$, whose coordinates are $\mathbb{Q}$-linearly independent. Further for $i = 1, ..., m+1$, let $q_i \in \overline{\mathbb{Q}}[X_1, ..., X_{\sum_{i=1}^m |\vec{p_i}|}]$. Then there is an $N(\vec{p_1}, ..., \vec{p_m}, q_1, ..., q_{m+1}) \in \mathbb{N}$ such that if $(a_1, ..., a_m) \in \mathbb{R}^m$ is such that*

$$q_i(a_1^{\vec{p_1}}, ..., a_m^{\vec{p_m}}) = 0, \text{ for all } i = 1, ..., m+1, \tag{3.1.1}$$

and

$$\mathfrak{Q}_{m+1,q_1,...,q_{m+1}}(a_1^{\vec{p_1}}, ..., a_m^{\vec{p_m}}) \neq 0, \tag{3.1.2}$$

then there exists, for every $i = 1, ..., m$ and every $j = 1, ..., |\vec{p_i}|$, elements $m_{i,j} \in \mathbb{Z}$, not all zero, such that

(iii) $|m_{i,j}| \leq N$ *for all* $i = 1, ..., m$ *and* $j = 1, ..., |\vec{p_i}|$, *and*

(iv) $\prod_{i=1}^m \prod_{j=1}^{|\vec{p_i}|} a_i^{m_{i,j} \cdot p_{i,j}} = 1$.

Using the argument which Kirby and Zilber developed in [KZ06], one can show that Condition 3.1.2 implies Condition 3.1.3. In the rest of this text, we need the following uniform version of Condition 3.1.2, which is not only uniform in elements satisfying given polynomial equations, but which is uniform in all elements which satisfy *some* polynomial equations in a fixed degree:

Condition 3.1.4 *Let $d \geq 1$, $m \in \mathbb{N}$ and for $i = 1, ..., m$ let $\vec{p_i} \in \mathbb{Q}(\tau)^{|\vec{p_i}|}$, whose coordinates are $\mathbb{Q}$-linearly independent. Then there is an $N(d, \vec{p_1}, ..., \vec{p_m}) \in \mathbb{N}$ such that if*

(i) $a_i \in \mathbb{R}$, *for* $i = 1, ..., m$, *and*

(ii) there is a polynomial $q_i \in \overline{\mathbb{Q}}[X_1, ..., X_{\sum_{i=1}^m |\vec{p_i}|}]$ *for* $i = 1, ..., m+1$, *where each* q_i *has degree at most* d,

such that

$$q_i(a_1^{\vec{p_1}}, ..., a_m^{\vec{p_m}}) = 0, \text{ for all } i = 1, ..., m+1, \tag{3.1.3}$$

and

$$\mathfrak{Q}_{m+1,q_1,...,q_{m+1}}(a_1^{\vec{p_1}}, ..., a_m^{\vec{p_m}}) \neq 0, \tag{3.1.4}$$

then there exists, for every $i = 1, ..., m$ and every $j = 1, ..., |\vec{p_i}|$, elements $m_{i,j} \in \mathbb{Z}$, not all zero, such that

(iii) $|m_{i,j}| \leq N$ *for all* $i = 1, ..., m$ *and* $j = 1, ..., |\vec{p_i}|$, *and*

(iv) $\prod_{i=1}^m \prod_{j=1}^{|\vec{p_i}|} a_i^{m_{i,j} \cdot p_{i,j}} = 1$.

Note that it is not known whether Condition 3.1.2 implies Condition 3.1.4. The method used in [KZ06] is not applicable to this case.

3.2 Schanuel condition for generic τ

In this section we show that Condition 3.1.4 actually holds for co-countably many elements τ of $\mathbb{R}$.

Theorem 3.2.1 *Let $\tau \in \mathbb{R}$, and assume τ is not 0-definable in $\mathbb{R}_{exp}$. Then Condition 3.1.4 holds for τ.*

The proof is based on the following theorem of Ax which is statement (SD) on page 253 of [A71].

Theorem 3.2.2 *Let F be a field and D a derivation on F with field of constants $C \supseteq \mathbb{Q}$. Let $y_1, ..., y_n, z_1, ..., z_n \in F^{\times}$ with*

(1) $Dy_i = \frac{Dz_i}{z_i}$, for $i = 1, ..., n$, and

(2) $(Dy_1, ..., Dy_n)$ linearly independent over $\mathbb{Q}$.

Then

$$tr.deg_C C(y_1, ..., y_n, z_1, ..., z_n) \geq n + 1.$$

The next lemma is a corollary of Ax's Theorem. The proof given is from the unpublished paper [W03] by Wilkie.

Lemma 3.2.3 *Let $\langle F, \delta \rangle$ be a differential field with fields of constants $C \supseteq \mathbb{Q}$ and let $m, r \geq 1, d \geq 0$. Suppose that*

(i) $b_1, ..., b_m$ are elements of F linearly independent over $\mathbb{Q}$,

(ii) H is a subfield of F closed under δ,

(iii) $H \cap C = \mathbb{Q}$,

(iv) $dim_H V = d$, where V is the H-vector space generated by $b_1, ..., b_m$ inside F,

(v) H_1 is any subfield of F containing both H and C with $tr.deg_C(H_1) = r$,

(vi) $e_1, ..., e_m$ are non-zero elements of F satisfying $\delta(e_i) = e_i \cdot \delta(b_i)$ for $i = 1, ..., m$.

Then

$$tr.deg_C C(e_1, ..., e_n) \geq m - d - r + 1. \tag{3.2.1}$$

Proof: Note that all the hypotheses and the conclusion are unaffected if we replace $\langle b_1, ..., b_m \rangle$ by $\langle b_1, ..., b_m \rangle A$ and the $e_1, ..., e_m$ by the corresponding multiplicative combinations, where A is any non-singular $m \times m$ matrix over $\mathbb{Q}$. Assume this has been done such that $q \geq 0$ is maximal such that

(vii) $b_1, ..., b_q \in C$.

This implies that

(viii) no non-trivial linear combination of $b_{q+1}, ..., b_m$ with $\mathbb{Q}$-coefficients lies in C.

Note that if $d \geq m$, we get the conclusion (3.2.1), since $r \geq 1$. So we can assume

(ix) $d < m$.

Now by (i), $b_1, ..., b_q$ are linearly independent over $\mathbb{Q}$. In fact, if $q > 0$, then

(x) $b_1, ..., b_q$ are linearly independent over H.

Proof: Suppose that for $1 \leq t < q$, $b_1, ..., b_t$ are linearly independent over H and there are $h_1, ..., h_t \in H$ such that

$$b_{t+1} = \sum_{i=1}^{t} h_i b_i.$$

By (viii), $b_1, ..., b_{t+1}$ are in C, and we have

$$0 = \delta(b_{t+1}) = \sum_{i=1}^{t} \delta(h_i) b_i.$$

By (ii), $\delta(h_1), ..., \delta(h_n) \in H$ and by the hypothesis on $b_1, ..., b_t$, we have $\delta(h_1) = ... = \delta(h_t) = 0$ and hence $h_1, ..., h_n \in C$. Thus (iii) implies that $h_1, ..., h_t \in \mathbb{Q}$. But then $b_{t+1} = \sum_{i=1}^{t} h_i b_i$ contradicts (i).

$\square$(x)

By reordering the $b_{q+1}, ..., b_m$, we may now assume that $b_1, ..., b_q, b_{q+1}, ..., b_d$ is a basis of V as an H-vector space.

Now we consider the subfield $K_0 := C(b_{q+1}, ..., b_m, e_{q+1}, ..., e_m)$ of F and try to compute an upper bound for $tr.deg_C K_0$. Therefore define $K_1 := C(b_{q+1}, ..., b_m)$. First note that $b_1, ..., b_q \in C$ and that $b_{d+1}, ..., b_m$ can be written as an H-linear combination of $b_1, ..., b_d$. Thus we get that

$$\begin{aligned} tr.deg_C K_1 &= tr.deg_C C(b_{q+1}, ..., b_m) \\ &\leq tr.deg_C H_1(b_{q+1}, ..., b_m) \\ &= tr.deg_C H_1(b_{q+1}, ..., b_d) \\ &\leq tr.deg_C H_1 + d - q = r + d - q. \end{aligned}$$

Suppose now for a contradiction that $tr.deg_C C(e_1, ..., e_m) \leq m - d - r$. Thus

$$\begin{aligned} tr.deg_C K_0 &= tr.deg_C K_1 + tr.deg_{K_1} K_0 \\ &\leq (r + d - q) + (m - d - r) = m - q \leq m. \end{aligned}$$

We now apply Theorem 3.2.2. This implies that $\delta(b_{q+1}), ..., \delta(b_m)$ are $\mathbb{Q}$-linearly dependent. This contradicts (viii).

□

Proof of Theorem 3.2.1: Let $\mathfrak{L}_\partial^d$ be the language of differential rings augumented by constant symbols

- ω,
- for every $i = 1, ..., m$ and $j = 1, ..., |\vec{p_i}|$: $b_{i,j}$ and $e_{i,j}$
- for every $i = 1, ..., m + 1$ and $\alpha \in \{0, ..., d\}^{\{1,...,M\}}$: $q_{i,\alpha}$,

where $M := \sum_{i=1}^m |\vec{p_i}|$.

In the following definition of T_d we will write q_i for the polynomial

$$\sum_{\alpha \in \{0,...,d\}^{\{1,...,M\}}} q_{i,\alpha} X_1^{\alpha(1)} \cdot ... \cdot X_M^{\alpha(M)}.$$

Further for $i = 1, ..., m$ and $j = 2, ..., |\vec{p_i}|$, let $r_{i,j} \in \mathbb{Q}(X)$ be such that $r_{i,j}(\tau) = \frac{p_{i,j}}{p_{i,1}}$. So let T_d be an $\mathfrak{L}_\partial^d$-theory defined by the following axioms:

(T_d1) Axioms for differential fields of characteristic zero,

(T_d2) $\partial\omega = 1$,

(T_d3) $b_{i,j} = r_{i,j}(\omega) \cdot b_{i,1}$, for $i = 1, ..., m$ and $j = 2, ..., |\vec{p_i}|$,

(T_d4) $\sum_{i=1}^m \sum_{j=1}^{n_i} m_{i,j} b_{i,j} \neq 0$, for $m_{i,j} \in \mathbb{Z}$, not all zero,

(T_d5) $\partial e_{i,j} = e_{i,j}\partial b_{i,j}$, for $i = 1, ..., m$ and $j = 1, ..., |\vec{p_i}|$,

(T_d6) $\partial q_{i,\alpha} = 0$, for every $i = 1, ..., m+1$ and $\alpha \in \{0, ..., d\}^{\{1,...,M\}}$

(T_d7) $q_i(e_{1,1}, ..., e_{m,|\vec{p_i}|}) = 0$, for $i = 1, ..., m+1$.

(T_d8) $\mathfrak{Q}_{M,q_1,...,q_m,q_{m+1}}(e_{1,1}, ..., e_{m,|\vec{p_i}|}) > 0$.

Now let $(F, \partial, ...) \models T_d$, where, with the usual abuse of notation, we denote the interpretation of the constants by the respective constants symbol. Let C be the field of constants of (F, ∂). Without loss of generality, we can assume that $\mathbb{Q} \subseteq C$. Set $H := \mathbb{Q}(\omega)$, $H_1 = C(\omega)$ and $r := 1$ and apply Lemma 3.2.3. This implies that

$$tr.deg_C C(e_{1,1}, ..., e_{m,|\vec{p_m}|}) \geq \sum_{i=1}^{m} |\vec{p_i}| - d,$$

where d is the H-linear dimension of the H-vector space generated by $b_{1,1}, ..., b_{m,|\vec{p_m}|}$. By (T_d3), we know that $d \leq m$. Hence the last equation gives

$$tr.deg_C C(e_{1,1}, ..., e_{m,|\vec{p_m}|}) \geq \sum_{i=1}^{m} |\vec{p_i}| - m.$$

This contradicts (T_d7) and (T_d8). Hence T_d is inconsistent. So there is a natural number $N = N(d)$ such that $T_{d,N}$ is inconsistent, where $T_{d,N}$ is the same the theory as T_d expect that (T_d4) is replaced by:

$(T_{d,N}4)$ $\sum_{i=1}^{m}\sum_{j=0}^{n_i} m_{i,j} b_{i,j} \neq 0$, for $m_{i,j} \in \mathbb{Z}$, not all zero and $|m_{i,j}| \leq N$.

We now want to show that N is the natural number we are searching for in the theorem. This means, we want to show that for any $(a_i \in \mathbb{R})_{i=1,...,m}$ and any $(q_i \in \overline{\mathbb{Q}}[X_1, ..., X_M])_{i=1,...,m+1}$ satisfying (3.1.3) and (3.1.4), there are $m_{i,j} \in \mathbb{Z}$, not all zero and $|m_{i,j}| \leq N$, such that

$$\prod_{i=1}^{m}\prod_{j=1}^{|\vec{p_i}|} a_i^{m_{i,j}\cdot p_{i,j}} = 1.$$

Therefore let $a_1, ..., a_m \in \mathbb{R}$ and $q_1, ..., q_n \in \overline{\mathbb{Q}}[X_1, ..., X_M]$ satisfy (3.1.3) and (3.1.4). Now it's only left to show that the assumptions of the theorem give us a model of T_d minus (T_d4). Since τ is not definable in $\mathbb{R}_{exp}$, we have that $dim\{\tau\} = 1$ in the pregeometry $\mathbf{cl}_{T_{exp}}$, where $T_{\exp}$ is the theory of $\mathbb{R}_{exp}$. Hence there exists a set $B \subset \mathbb{R}$ such that $\{\tau\} \cup B$ is a basis for the pregeometry $\mathbf{cl}_{T_{exp}}$. Let C be $\mathbf{cl}_{T_{exp}}(B)$, the model generated of T_{exp} generated by B. Let $\mathfrak{L}_e$ be the language of $\mathbb{R}_{exp}$ and let

$\mathfrak{L}_{e,C}$ be the language $\mathfrak{L}_e$ plus a constant symbol for every $c \in C$. Since $\{\tau\} \cup B$ is a basis, $\tau \notin C$, but for any $a \in \mathbb{R}$ there exists an $\mathfrak{L}_{e,C}$-0-definable function $\chi : \mathbb{R} \to \mathbb{R}$ such that $\chi(\tau) = a$. Since $\mathbb{R}_{exp}$ is o-minimal, by Theorem 2.1.3, the function $\chi(x)$ is differentiable at all but finitely many points. Since χ is $\mathfrak{L}_e$-definable with parameters in C, these finitely many points lie in C. Hence χ is differentiable on an open interval around τ. Further suppose there is another $\mathfrak{L}_{e,C}$-0-definable function $\xi : \mathbb{R} \to \mathbb{R}$ with $\xi(\tau) = a$. By o-minimality, the boundary of the set $\{x \in \mathbb{R} \mid \chi(x) = \xi(x)\}$ is finite and $\mathfrak{L}_{e,C}$-0-definable. Hence $\chi(x)$ and $\xi(x)$ agree on an open interval around τ. The same statement also holds for the derivates of ξ and χ. Thus the function $\delta : \mathbb{R} \to \mathbb{R}$, which sends an element $a \in \mathbb{R}$ to $\frac{d\chi}{dx}(\tau)$ where $\chi : \mathbb{R} \to \mathbb{R}$ is any $\mathfrak{L}_{e,C}$-0-definable function with $\chi(\tau) = a$, is well-defined.

(1) δ is a derivation on the field $\mathbb{R}$ with field of constants C.

Proof: Let $a, b \in \mathbb{R}$ and let ξ, χ be $\mathfrak{L}_{e,C}$-0-definable functions with $\xi(\tau) = a$ and $\chi(\tau) = b$. Then the function $\sigma(-) := \chi(-) \cdot \xi(-)$ is $\mathfrak{L}_{e,C}$-0-definable with $\sigma(\tau) = a \cdot b$. By the Leibniz-rule,

$$\delta(ab) = \frac{d\sigma}{dx}(\tau) = \frac{d\chi}{dx}(\tau) \cdot \xi(\tau) + \frac{d\xi}{dx}(\tau) \cdot \chi(\tau) = \delta(a)b + \delta(b)a.$$

Since all elements of C are definable by a constant $\mathfrak{L}_{e,C}$-0-definable function, $\delta(C) = 0$. Hence δ is a derivation whose field of constants contains C. In fact, let $\alpha \in \mathbb{R}$ with $\delta(\alpha) = 0$. Let χ be an $\mathfrak{L}_{e,C}$-0-definable function, with $\chi(\tau) = \alpha$ and $\frac{d\chi}{dx}(\tau) = 0$. Hence by o-minimality, there are $b_1, b_2 \in C$ such that $\tau \in (b_1, b_2)$ and $\chi(\frac{b_1+b_2}{2}) = \alpha$. Hence $\alpha \in C$. Thus the field of constants of δ is C.

$\square(1)$

So (T_d1) is satisfied by $(\mathbb{R}, \delta)$. Since $\delta(\tau) = 1$, (T_d2) is also satisfied by $(\mathbb{R}, \delta, \tau)$.
Now let $b_{i,j} := p_{i,j} \log(a_i)$ for $i = 1, ..., m$ and $j = 1, ..., |\vec{p_i}|$. This definition implies (T_d3), if we interpret the $b_{i,j}$'s as respective constants in the structure $(\mathbb{R}, \delta, \tau)$. Further set $e_{i,j} := exp(b_i)$. In fact, $e_{i,j} = a_i^{p_{i,j}}$. If we also interpret the $e_{i,j}$'s as respective constants, we can satisfy (T_d5). Finally by interpreting the coefficient of the $q_1, ..., q_{m+1}$ as the constants of the form $q_{i,\alpha}$, our assumptions (3.1.3) and (3.1.4) imply (T_d7) and (T_d8). Since the $q_1, ..., q_{m+1}$ are polynomials over $\overline{\mathbb{Q}}$ and $\overline{\mathbb{Q}} \subseteq C$, we also satisfy (T_d6). By the above, this model does not satisfy $(T_{d,N}4)$. Hence there are $m_{i,j} \in \mathbb{Z}$, not all zero and $|m_{i,j}| \leq N$, such that

$$\sum_{i=1}^{m} \sum_{j=1}^{|\vec{p_i}|} m_{i,j} b_{i,j} = 0.$$

Since $b_{i,j} = \log(a_i^{p_{i,j}})$, we finally have

$$\prod_{i=1}^{m} \prod_{j=1}^{|\vec{p_i}|} a_i^{m_{i,j} \cdot p_{i,j}} = 1.$$

□

3.3 Mann property

In this section, we consider the second main tool which is used in this text: the Mann property. Let F be a field and G be any subgroup of the multiplicative group $F^{\times}$. Consider equations of the form

$$a_1 x_1 + ... + a_n x_n = 1,$$

where $a_1, ..., a_n \in \mathbb{Q}$. We say a solution $(b_1, ..., b_n) \in G^n$ is *non-degenerate* if for every non-empty subset I of $\{1, ..., n\}$, $\sum_{i \in I} a_i b_i \neq 0$. Further we say that G has the *Mann property* if every equation of the above type has only finitely many non-degenerate solutions in G^n. We also call these solutions *Mann solutions*. In fact, it follows from work of Evertse in [E84] and van der Poorten and Schlickewei in [PS91] that

Theorem 3.3.1 *Every multiplicative subgroup of finite rank has the Mann property.*

See also [ESS02] Theorem 1.1 for a uniform bound on the number of non-degenerate solutions in the subgroup. The *rank* of an abelian group G is the dimension of the $\mathbb{Q}$-vector space $G \otimes \mathbb{Q}$, where G and $\mathbb{Q}$ are viewed as $\mathbb{Z}$-modules. Examples of groups with finite rank are $2^{\mathbb{Z}}, 2^{\mathbb{Q}}, 2^{\mathbb{Z}}3^{\mathbb{Z}}$ or $2^{\mathbb{Q}} \cdot (2^{\tau})^{\mathbb{Q}}$. Another well known example is the multiplicative subgroup $\mathbb{U}$ of roots of unity in the field of complex numbers.
Now consider a homogenous equation

$$a_1 x_1 + ... + a_n x_n = 0, \tag{3.3.1}$$

with $a_1, ..., a_n \in \mathbb{Q}$. Again we say a solution $(b_1, ..., b_n) \in G^n$ is non-degenerate if for every non-empty subset I of $\{1, ..., n\}$ $\sum_{i \in I} a_i b_i \neq 0$. We now describe the set of non-degenerate solutions of equation (3.3.1). Assuming $a_n \neq 0$, let S be the set of non-degenerate solutions of

$$-\frac{a_1}{a_n} y_1 - ... - \frac{a_{n-1}}{a_n} y_{n-1} = 1.$$

Then the set of non-degenerate solutions of equation (3.3.1) is

$$\bigcup_{(b_1,...,b_{n-1}) \in S} (b_1, ..., b_{n-1}, 1)G.$$

This fact directly implies the following statement.

Proposition 3.3.2 *Let $H \subset G$ be two groups with the Mann property. Suppose that all Mann solutions of G are already in H. Then for every non-degenerate $(g_1, ..., g_n) \in G$ satisfying an equation of the form* (3.3.1), *there are $h_1, ..., h_n \in H$ and $g \in G$ such that for $i = 1, ..., n$*

$$g_i = h_i \cdot g.$$

3.4 Schanuel condition implies Mann property

In this section we will show a connection between the Schanuel condition 3.1.2 and the Mann property.

Definition 3.4.1 *Let Γ be a multiplicative subgroup of $\mathbb{R}$. For $p \in \mathbb{Q}(\tau)$, the* subgroup of p-powers of Γ *is defined by*

$$\Gamma^{[p]} := \{\gamma^p \mid \gamma \in \Gamma\}.$$

The $\mathbb{Q}(\tau)$-closure of Γ *is defined by*

$$\Gamma^{[\mathbb{Q}(\tau)]} := \{\gamma_1^{p_1} \cdot ... \cdot \gamma_n^{p_n} \mid n \in \mathbb{N}, \gamma_i \in \Gamma, p_i \in \mathbb{Q}(\tau)\}.$$

Lemma 3.4.2 *Let Γ be a multiplicative subgroup of $\mathbb{R}$ of finite rank. Then for every $(p_1, ..., p_n) \in \mathbb{Q}(\tau)^n$, the group $\Gamma^{[p_1]} \cdot \Gamma^{[p_2]} \cdot ... \cdot \Gamma^{[p_n]}$ has the Mann property.*

<u>Proof:</u> Since Γ is of finite rank, for every $(p_1, ..., p_n) \in \mathbb{Q}(\tau)^n$ the group $\Gamma^{[p_1]} \cdot \Gamma^{[p_2]} \cdot ... \cdot \Gamma^{[p_n]}$ has finite rank as well. Thus this group has the Mann property by Theorem 3.3.1.

□

Corollary 3.4.3 *Assume τ is algebraic. Let Γ be a multiplicative subgroup of $\mathbb{R}$ of finite rank, then $\Gamma^{[\mathbb{Q}(\tau)]}$ has the Mann property.*

<u>Proof:</u> This directly follows from Lemma 3.4.2 and the fact that if τ is algebraic of degree d, then

$$\Gamma^{[\mathbb{Q}(\tau)]} = \Gamma \cdot \Gamma^{[\tau]} \cdot ... \cdot \Gamma^{[\tau^{d-1}]}.$$

□

An obvious question is now whether this also holds for non-algebraic τ. It turns out that this is actually the case if the group Γ is algebraic enough and τ satisfies Schanuel condition 3.1.2.

Definition 3.4.4 *Let* Γ *be a multiplicative subgroup of* $\mathbb{R}$*. We say* Γ *is* point-wise algebraic *if every* $g \in \Gamma$ *is algebraic.*

Theorem 3.4.5 *Assume Condition 3.1.2 holds for* τ*. Let* Γ *be a multiplicative subgroup of* $\mathbb{R}$*. If* Γ *is point-wise algebraic and has the Mann property, every Mann solution in* $\Gamma^{[\mathbb{Q}(\tau)]}$ *is in* Γ *and thus* $\Gamma^{[\mathbb{Q}(\tau)]}$ *has the Mann property.*

Proof: Let $a_1, ..., a_n \in \mathbb{Q}$, $\vec{p_1}, ..., \vec{p_n}$ be tuples with coordinates in $\mathbb{Q}(\tau)$ and $\vec{g_1}, ..., \vec{g_n}$ be tuples with coordinates in Γ with $|\vec{g_i}| = |\vec{p_i}|$ for $i = 1, ..., n$. We write

$$\vec{g_i}^{\vec{p_i}} := \prod_{j=1}^{|\vec{g_i}|} g_{i,j}^{p_{i,j}}.$$

Suppose that

$$a_1\vec{g_1}^{\vec{p_1}} + ... + a_n\vec{g_n}^{\vec{p_n}} = 1 \tag{3.4.1}$$

and for all $I \subseteq \{1, ..., n\}$

$$\sum_I a_i\vec{g_i}^{\vec{p_i}} \neq 0. \tag{3.4.2}$$

Further take $\{t_1, ..., t_m\}$ be a $\mathbb{Q}$-linearly independent subset of

$$\{1, p_{1,1}, ...p_{1,|\vec{p_1}|}, p_{2,1}, ..., p_{n,|\vec{p_n}|}\}$$

with $t_1 = 1$ and set $\vec{t} := (t_1, ..., t_m)$. If $m = 1$, we have $\vec{g_i}^{\vec{p_i}} \in \Gamma$ for $i = 1, ..., n$ and $j = 1, ..., |\vec{g_i}|$. So it is just left to show that $m = 1$.
For a contradiction, suppose that $m > 1$. Let $h_1, ..., h_k$ be a maximal subset of

$$\{g_{i,j} \mid i = 1, ..., n,\ j = 1, ..., |\vec{g_i}|\},$$

such that $h_1^{\vec{t}}, ..., h_k^{\vec{t}}$ are multiplicatively independent. Hence we have for $j = 1, ..., n$

$$\vec{g_j}^{\vec{p_j}} = \prod_{i=1}^{k}\prod_{r=1}^{m} h_i^{\alpha_{i,j,r}\cdot t_r},$$

where $\alpha_{i,j,r} \in \mathbb{Z}$. Then we get a rational function $q \in \overline{\mathbb{Q}}(X_{1,1}, ..., X_{k,m})$ defined by

$$\sum_{j=1}^{n} a_j \cdot \prod_{i=1}^{k}\prod_{r=1}^{m} X_{i,r}^{\alpha_{i,j,r}} - 1.$$

Since $m > 1$, there is $r > 1$ such that there is $i \in \{1, ..., k\}$ and $j \in \{1, ..., m\}$ with $\alpha_{i,j,r} \neq 0$. Hence q is not in the subfield $\overline{\mathbb{Q}}(X_{1,1}, ..., X_{k,1})$, which is the field of rational functions in variables $X_{i,1}$ where $i = 1, ..., k$. By equation (3.4.1), q vanishes at $(h_1^{\vec{t}}, ..., h_k^{\vec{t}})$.

(1) q does not vanish everywhere.

Proof: If q vanishes everywhere, then $q = 0$. Since not all $\alpha_{i,j,r}$ are zero, there is $J \subseteq \{1, ..., n\}$ such that

$$a_j \cdot \prod_{i=1}^{k} \prod_{r=1}^{m} X_{i,r}^{\alpha_{i,j,r}} \text{ is non-constant for every } j \in J,$$

and further

$$\sum_{j=1}^{n} a_j \cdot \prod_{i=1}^{k} \prod_{r=1}^{m} X_{i,r}^{\alpha_{i,j,r}}$$

is identically zero. This implies that

$$\sum_{j \in J} a_j \vec{g_j}^{\vec{p_j}} = 0.$$

This contradicts inequality (3.4.2). Thus q does not vanish everywhere.

□(1)

By (1) q is a non-trivial element of $\overline{\mathbb{Q}}(X_{1,1}, ..., X_{k,m}) - \overline{\mathbb{Q}}(X_{1,1}, ..., X_{k,1})$. Since $h_i \in \overline{\mathbb{Q}}$, this implies that

$$tr.deg_{\mathbb{Q}}(h_1^{\vec{t}}, ..., h_k^{\vec{t}}) < k|\vec{t}| - k.$$

Hence by Condition 3.1.2 $h_1^{\vec{t}}, ..., h_k^{\vec{t}}$ are multiplicatively dependent. This is a contradiction against our assumptions about $h_1^{\vec{t}}, ..., h_k^{\vec{t}}$. Hence $m = 1$.

□

3.5 Regularly dense groups

We will revise some results on regularly dense groups which Robinson and Zakon proved in [RZ60]. See also [vdDG06] p. 71.

Definition 3.5.1 *An ordered abelian group G is* regularly dense, *if $G \neq \{1\}$ and for every $g, h \in G$ with $g < h$ and every $n \in \mathbb{N}_{>0}$, there is a $k \in G$ with $g < k^n < h$.*

Note that for an abelian group G being regularly dense means that for every $n \in \mathbb{N}_{>0}$ the subgroup $G^{[n]}$ is dense in G. First note the following easy fact about subgroups in real closed fields.

Proposition 3.5.2 *Let R be a real closed field and G a dense multiplicative subgroup of $R_{>0}$. Then G is regularly dense.*

Proof: Take $g, h \in G$. Since $g, h > 0$ and R real closed, we know that $g^{\frac{1}{n}}, h^{\frac{1}{n}} \in R$. Since G is dense in $R_{>0}$, there is $k \in G \cap (g^{\frac{1}{n}}, h^{\frac{1}{n}})$. Hence $g < k^n < h$.

□

Proposition 3.5.3 *Let G, G' be regularly dense ordered groups such that for every $n \in \mathbb{N}_{>0}$*

$$|G : G^{[n]}| = |G' : G'^{[n]}|.$$

Let H be a pure subgroup of G, H' be a pure subgroup of G' and $\beta : H \to H'$ be an ordered group isomorphism. Further suppose that G' is $|H|^+$-saturated. Then for every $g \in G - H$, there exists an $h \in G' - H'$ such that for all $k \in H, m \in \mathbb{Z}$ and $n \in \mathbb{N}_{>0}$,

(i) $k < g^n$ iff $\beta(k) < h^n$, and

(ii) $kg^m \in G^{[n]}$ iff $\beta(k)h^m \in G'^{[n]}$.

Further note that in the situation of Proposition 3.5.3, the isomorphism β extends to an isomorphism γ from the divisible closure of H and g in G to the divisible closure of H' and g' in G' sending g to g'.

Let $\mathfrak{L}_g$ be the language $(<, 1, \cdot, ^{-1})$, the language of the theory of ordered groups. Further, let T_G be the $\mathfrak{L}_g$-theory of regularly dense ordered abelian groups G with fixed

$$|G : G^{[n]}| = \varrho(n) < \infty.$$

Further let $\mathfrak{L}_g^+$ be the language $\mathfrak{L}_g$ augmented by a new unary predicate D_n for every $n \in \mathbb{N}_{>0}$. Then define T_G^+ to be the $\mathfrak{L}_g^+$ -theory consisting of the axioms of T_G and the following axioms: for every $n \in \mathbb{N}_{>0}$,

$$\forall x(D_n(x) \leftrightarrow \exists y(x = y^n)).$$

Theorem 3.5.4 *[RZ60] $(T_G)^+$ admits elimination of quantifiers.*

Robinson and Zakon only proved model completeness in [RZ60], but van den Dries and Günaydin noted in [vdDG06] Lemma 7.7 that the Robinson and Zakons construction directly gives the stated quantifier elimination result.

Chapter 4

Groups orthogonal to the power functions

In this chapter we will analyze the structure $(\tilde{\mathbb{R}}, \Gamma)$, where Γ is orthogonal to the power functions definable in $\tilde{\mathbb{R}}$. To be precise, we consider *dense* multiplicative subgroups of $\mathbb{R}$ satisfying

(G1) every $g \in \Gamma$ is algebraic over $\mathbb{Q}$,

(G2) $|\Gamma : \Gamma^{[n]}| < \infty$ for $n \in \mathbb{N}_{>0}$,

(G3) for every $\mathbb{Q}$-linearly independent $(p_1, ..., p_n) \in \mathbb{Q}(\tau)^n$, the group $\Gamma^{[p_1]} \cdot \Gamma^{[p_2]} ... \cdot \Gamma^{[p_n]}$ has the Mann property,

(G4) for $n \geq 2$, for every $\mathbb{Q}$-linearly independent $(p_1, ..., p_n) \in \mathbb{Q}(\tau)^n$,

$$\Gamma^{[p_1]} \cap \Gamma^{[p_2]} \cdot ... \cdot \Gamma^{[p_n]} = \{1\},$$

where for every $p \in \mathbb{Q}(\tau)$, $\Gamma^{[p]}$ is the set $\{g^p | g \in \Gamma\}$. Note that (G1) implies that Γ is countable.

4.1 Groups satisfying (G1)-(G4)

In this section we show which kind of groups satisfy (G1)-(G4). Especially we will prove that $2^{\mathbb{Q}}$ is one of these groups.

Lemma 4.1.1 *Let Γ be a multiplicative subgroup of $\mathbb{R}$ of finite rank. Then Γ satisfies condition (G3).*

Proof: Since Γ is of finite rank, for every $\mathbb{Q}$-linearly independent $(p_1, ..., p_n) \in \mathbb{Q}(\tau)^n$ the group $\Gamma^{[p_1]} \cdot \Gamma^{[p_2]} \cdot ... \cdot \Gamma^{[p_n]}$ has finite rank. Thus this group has the Mann property by Theorem 3.3.1.

□

Proposition 4.1.2 *Let $\tau \in \mathbb{R} - \mathbb{Q}$, then the group $2^{\mathbb{Q}}$ satisfies conditions (G1)-(G4).*

Proof: First note that conditions (G1) and (G2) are obviously satisfied and condition (G3) follows from Lemma 4.1.1, since $2^{\mathbb{Q}}$ is of rank 1. Now for condition (G4), let $(p_1, ..., p_n)$ be a $\mathbb{Q}$-linearly independent tuple in $\mathbb{Q}(\tau)^n$. Suppose $(2^m)^{p_n} \in (2^{\mathbb{Q}})^{[p_1]} \cdot (2^{\mathbb{Q}})^{[p_2]} \cdot ... \cdot (2^{\mathbb{Q}})^{[p_{n-1}]}$. Then there are $l_1, ..., l_{n-1} \in \mathbb{Q}$ such that

$$(2^m)^{p_n} = (2^{l_1})^{p_1} \cdot ...(2^{l_{n-1}})^{p_{n-1}}.$$

This directly implies that

$$0 = p_n \cdot m \cdot \log(2) - p_{n-1} l_{n-1} \log(2) - ... - p_1 \cdot l_1 \cdot log(2).$$

and thus

$$0 = p_n \cdot m - p_{n-1} l_{n-1} - ... - p_1 \cdot l_1.$$

Since $(p_1, ..., p_n)$ are $\mathbb{Q}$-linearly independent, we get $m = l_1 = ... = l_{n-1} = 0$. Hence $2^m = 1$. Thus $2^{\mathbb{Q}}$ satisfies also condition (G4).

□

Note that the the group $2^{\mathbb{Z}} 3^{\mathbb{Z}}$ is finite rank, but still is not suitable for $\tau = log_2(3)$. The equation $2^\tau = 3$ contradicts condition (G4).

Lemma 4.1.3 *Let* Γ *be a multiplicative subgroup of* $\mathbb{R}$. *If* $\mathbb{Q}(\tau)$ *is linearly disjoint from* $\mathbb{Q}(\log(\Gamma))$ *over* $\mathbb{Q}$, *then* Γ *satisfies condition (G4).*

Proof: Let $(p_1, ..., p_n)$ be a $\mathbb{Q}$-linearly independent tuple in $\mathbb{Q}(\tau)^n$ and $g \in \Gamma$. For a contradiction suppose that $(g)^{p_n} \in \Gamma^{[p_1]} \cdot ... \cdot \Gamma^{[p_{n-1}]}$ and $g \neq 1$. Then there are $h_1, ..., h_{n-1} \in \Gamma$ such that

$$g^{p_n} = h_1^{p_1} \cdot ... h_{n-1}^{p_{n-1}}.$$

This directly implies that

$$0 = p_n \cdot \log(g) - p_{n-1} \log(h_1) - ... - p_1 \cdot log(h_{n-1}).$$

Since $\log(g) \neq 0$, we get that $(p_1, ..., p_n)$ is linearly dependent over $\mathbb{Q}(\log(\Gamma))$. Since $\mathbb{Q}(\tau)$ is linearly disjoint from $\mathbb{Q}(\log(\Gamma))$ over $\mathbb{Q}$, $(p_1, ..., p_n)$ are linearly dependent over $\mathbb{Q}$. This is a contradiction to our assumptions. Hence Γ satisfies condition (G4).

□

Corollary 4.1.4 *Let* $\tau \in \mathbb{R}$ *with* $\mathbb{Q}(\tau)$ *linearly disjoint from* $\mathbb{Q}(\log(2), \log(3))$ *over* $\mathbb{Q}$. *Then* $2^{\mathbb{Z}} 3^{\mathbb{Z}}$ *satisfies (G1)-(G4).*

4.2 Schanuel conditions for the group

For the following, we fix a multiplicative subgroup Γ of $\mathbb{R}$. Recall from page 5 that $\mathfrak{L}_\Gamma^\tau(G)$ is the language $\mathfrak{L}^\tau$ plus an extra unary predicate G and one constant symbol $\dot{\gamma}$ for every $\gamma \in \Gamma$.

Definition 4.2.1 *An $\mathfrak{L}_\Gamma^\tau(G)$-formula $\varphi(x_1, ..., x_m)$ is of* Schanuel-type *if there are $l \in \mathbb{N}$ with $l < m$ and*

(i) there is a $\mathbb{Q}$-linearly independent tuple $\vec{p} \in \mathbb{Q}(\tau)^{|\vec{p}|}$

(ii) there is $q_i \in \mathbb{Q}[X_1, ..., X_{m|\vec{p}|}]$, for $i = 1, ..., m - l + 1$,

such that $\varphi(x_1, ..., x_m)$ is the conjunction of the following formulas

$$\begin{aligned} q_1(x_1^{\vec{p}}, ..., x_m^{\vec{p}}) &= 0, \\ \vdots \quad & \quad \vdots \\ q_{m-l+1}(x_1^{\vec{p}}, ..., x_m^{\vec{p}}) &= 0, \\ \bigwedge_{i=1}^{l} G(x_i)&, \end{aligned}$$

and

$$\mathfrak{Q}_{(m-l)|\vec{p}|, q_{1,\vec{y}}, ..., q_{m-l+1,\vec{y}}}(x_{l+1}^{\vec{p}}, ..., x_m^{\vec{p}}) \neq 0, \tag{4.2.1}$$

where

$$\vec{y} := (x_1^{\vec{p}}, ..., x_l^{\vec{p}}).$$

For the definition of $\mathfrak{Q}$ see Chapter 2. For a given Schanuel-type formula φ, we write $\vec{p}^\varphi, l_\varphi, q_i^\varphi$ for the objects witnessing that φ is of Schanuel-type.
The following proposition will show that every tuple of elements satisfying a Schanuel formula φ will be multiplicatively dependent. The multiplicative dependency will hold uniformly and hence this property is first-order expressible. In fact, this property is axiom (A6) from the introduction.

Proposition 4.2.2 *Assume τ satisfies Condition 3.1.4 and Γ satisfies (G1). Let $\varphi(x_1, ..., x_m)$ be a formula of Schanuel-type. Then there is a natural number $N(\varphi)$ such that for $a_1, ..., a_n \in \mathbb{R}$, if $(\tilde{\mathbb{R}}, \Gamma) \models \varphi(a_1, ..., a_m)$, then there are $m_{i,j} \in \mathbb{Z}$, not all zero and $|m_{i,j}| \leq N(\varphi)$ such that*

$$\prod_{i=1}^{m} \prod_{j=1}^{|\vec{p}^\varphi|} a_i^{m_{i,j} \cdot p_j^\varphi} = 1. \tag{4.2.2}$$

Proof: This is a direct corollary of Condition 3.1.4. Let $\varphi(x_1, ..., x_m)$ be a formula of Schanuel-type and d be the maximal degree of the $\{q_i^{\varphi} | i = 1, ..., m - l_{\varphi} + 1\}$. Set $N(\varphi)$ to be $N(d, (\vec{p}^{\varphi})_{i=1,...,m})$ given by Condition 3.1.4. It is now only left to show that $N(\varphi)$ satisfies the conclusion of the proposition. Therefore let $a_1, ..., a_m \in \mathbb{R}$ with $(\tilde{\mathbb{R}}, \Gamma) \models \varphi(a_1, ..., a_m)$. Since φ is of Schanuel-type, this implies that $a_1, ..., a_{l_{\varphi}} \in \Gamma$. Since Γ satisfies (G1), every element in Γ is algebraic. Hence for every $i = 1, ..., l_{\varphi}$, the polynomial

$$r_i := X_{s(i)} - a_i, \text{ where } s(i) = \begin{cases} 1, & \text{if i=1;} \\ (i-1)|\vec{p}^{\varphi}| + 1, & \text{otherwise.} \end{cases}$$

is a polynomial over $\overline{\mathbb{Q}}$ and $r_i(a_1^{\vec{p}^{\varphi}}, ..., a_m^{\vec{p}^{\varphi}}) = 0$. If the inequality

$$\mathfrak{Q}_{m|\vec{p}^{\varphi}|, r_1, ..., r_{l_{\varphi}}, q_1^{\varphi}, ..., q_{m-l_{\varphi}+1}^{\varphi}}(a_1^{\vec{p}^{\varphi}}, ..., a_m^{\vec{p}^{\varphi}}) \neq 0 \tag{4.2.3}$$

holds, we can apply Condition 3.1.4 on $d, (\vec{p}^{\varphi})_{i=1}^m$ and get the desired $m_{i,j} \in \mathbb{Z}$, not all zero and $|m_{i,j}| \leq N(\varphi)$, such that equation (4.2.2) holds. Thus it is only left to prove (4.2.3). To see this, note that $\frac{\partial r_i}{\partial X_j} = 0$ iff $i \neq s(i)$. This implies that the matrix

$$B := J_{m|\vec{p}^{\varphi}|, r_1, ..., r_{l_{\varphi}}, q_1^{\varphi}, ..., q_{m-l_{\varphi}+1}^{\varphi}}(a_1^{\vec{p}^{\varphi}}, ..., a_m^{\vec{p}^{\varphi}})$$

has an $(l_{\varphi} + (m - l_{\varphi})|\vec{p}|) \times (l_{\varphi} + (m - l_{\varphi})|\vec{p}|)$ submatrix of the form

$$\begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & \\ * & \dots & * & A \end{pmatrix} \tag{4.2.4}$$

where A is the matrix

$$J_{(m-l_{\varphi})|\vec{p}^{\varphi}|, q_{1,\vec{a}}^{\varphi}, ..., q_{m-l_{\varphi}+1,\vec{a}}^{\varphi}}(a_{l_{\varphi}+1}^{\vec{p}^{\varphi}}, ..., a_m^{\vec{p}^{\varphi}}).$$

and

$$\vec{a} := (a_1^{\vec{p}^{\varphi}}, ..., a_{l_{\varphi}}^{\vec{p}^{\varphi}}).$$

Since φ is of Schanuel-type, condition (4.2.1) implies that A has an $(m - l_{\varphi} + 1) \times (m - l_{\varphi} + 1)$ submatrix with non-zero determinant. Thus by (4.2.4), the matrix B has a $(m+1) \times (m+1)$ submatrix with non-zero determinant. Hence we get equation (4.2.3) by definition of $\mathfrak{Q}$.

□

4.3 Axioms dmG_Γ

For this section, let Γ be a dense multiplicative subgroup of $\mathbb{R}$ which satisfies (G1)-(G4). We can now define the first order axioms for dense multiplicative subgroups dmG_Γ in the language $\mathfrak{L}_\Gamma^\tau(G)$. In the definition of dmG_Γ we will use the following two abbreviations. For every $(p_1,..,p_n) \in \mathbb{Q}(\tau)^n$, let

$$(G^{[p_1]} \cdot G^{[p_2]} \cdot ... \cdot G^{[p_n]})(x) := \exists y_1 ... \exists y_n (\bigwedge_{i=1}^{n} G(y_i) \wedge x = \prod_{i=1}^{n} y_i^{p_i}).$$

Further for all $m \in \mathbb{N}$ and $\vec{\alpha} \in \mathbb{Q}^m$, and for each tuple $\vec{p} := (p_1,...,p_n) \in \mathbb{Q}(\tau)^n$, let $MS(\vec{p},\vec{\alpha})$ be the finite set of non-degenerate solutions in $\Gamma^{[p_1]} \cdot ... \cdot \Gamma^{[p_n]}$ of the equation

$$\alpha_1 x_1 + ... + \alpha_m x_m = 1.$$

Definition 4.3.1 *We define the $\mathfrak{L}_\Gamma^\tau(G)$-theory dmG_Γ by the following set of axioms:*

(dmG1) $\forall x > 0 \forall y > 0((x < y) \rightarrow (\exists z(G(z) \wedge x < z < y)))$,

(dmG2) $\forall x \forall y((G(x) \wedge G(y) \rightarrow G(x \cdot y))$,

(dmG3) $\forall x \forall y((G(x) \wedge x \cdot y = 1) \rightarrow G(y))$,

(dmG4$_\gamma$) $G(\dot{\gamma})$, $\dot{\gamma}\dot{\delta} = (\dot{\delta\gamma})$, $\dot{1} = 1$, $\dot{\gamma}(\dot{\gamma^{-1}}) = 1$, *for all* $\gamma, \delta \in \Gamma$,

(dmG5$_{\gamma,n}$) $\neg G^{[n]}(\dot{\gamma})$, *for all* $\gamma \in \Gamma$ *and* $n \in \mathbb{N}$ *with* $\gamma \notin \Gamma^{[n]}$,

(dmG6$_n$) $\exists y_1 ... \exists y_m (\bigwedge_{i=1}^m G(y_i)) \wedge (\bigwedge_{i,j \in \{1,...,m\}, i \neq j} \forall x \ G(x) \wedge x^n \neq y_i y_j^{-1}) \wedge \forall x(G(x) \rightarrow (\bigvee_{i=1}^m \exists z G(x) \wedge z^n = x y_i^{-1}))$, *for all* $n \in \mathbb{N}$ *with* $m := |\Gamma : \Gamma^{[n]}|$,

(dmG7$_{\vec{p}}$) $\forall x(G^{[p_1]}(x) \wedge (G^{[p_2]} \cdot ... \cdot G^{[p_{n-1}]})(x^{p_n})) \rightarrow x = 1)$, *for every* $\mathbb{Q}$*-linear tuple* $\vec{p} := (p_1,...,p_n) \in \mathbb{Q}(\tau)^n$,

(dmG8$_{\vec{p},m,\vec{\alpha}}$) $\forall x_1 ... \forall x_m ((\bigwedge_{i=1}^m (G^{[p_1]} \cdot ... \cdot G^{[p_n]})(x_i) \wedge \sum_{i=1}^m \alpha_i x_i = 1 \wedge \bigwedge_{D \subset 2^m} \sum_{i \in D} \alpha_i x_i \neq 0) \rightarrow (\bigvee_{(\gamma_1,...,\gamma_m) \in MS(\vec{p},\vec{\alpha})} \bigwedge_{i=1}^m x_i = \gamma_i))$, *for all* $m \in \mathbb{N}$ *and* $\vec{\alpha} \in \mathbb{Q}^m$, *and for all tuple* $\vec{p} := (p_1,...,p_n) \in \mathbb{Q}(\tau)^n$,

(dmG9$_\varphi$) $\varphi(x_1,...,x_m) \rightarrow \bigvee_{k_{1,1}=1}^{N_\varphi} ... \bigvee_{k_{m,|\vec{p_m}^\varphi|}=1}^{N_\varphi} \prod_{i=1}^m \prod_{j=1}^{|\vec{p}^\varphi|} x_i^{k_j \cdot p_j^\varphi} = 1$, *for all* $\mathfrak{L}_\Gamma^\tau(G)$*-formulas* φ *of Schanuel-type,*

(dmG10$_{d,\overline{f}}$) $\forall \overline{y} \forall a \forall b \exists z \ a < z < b \wedge (\forall \overline{x}(G(\overline{x}) \rightarrow (\bigwedge_{i=1}^d f_i(\overline{x},\overline{y}) \neq z)))$, *for all* $d \in \mathbb{N}$, *and* $\mathfrak{L}_\Gamma^\tau$*-0-definable functions* $f_i(\overline{x},\overline{y})$, $i = 1,...,d$.

Note that (dmG1), (dmG2), (dmG3) and (dmG4) force the interpretation of G to be a dense multiplicative subgroup of $\mathbb{R}$ with subgroup Γ. Further (dmG5) gives us that Γ is pure in the interpretation of G. The axiom (dmG6) implies that $|\Gamma : \Gamma^{[n]}| = |G : G^{[n]}|$, when G is the interpretation of the predicate G. Further axiom (dmG7) forces G to be orthogonal to all power functions definable in $\tilde{\mathbb{R}}$. The axiom (dmG8) states that for every $(p_1, ..., p_n) \in \mathbb{Q}(\tau)^n$ all Mann solutions of $G^{[p_1]} \cdot G^{[p_2]} \cdot ... \cdot G^{[p_n]}$ are actually in the subgroup $\Gamma^{[p_1]} \cdot ... \cdot \Gamma^{[p_n]}$. Then axiom (dmG9) forces solutions of formulas of Schanuel-type to be multiplicatively dependent. Finally axiom (dmG10) is the statement that for a finite number of $\mathfrak{L}^{\tau}_{\Gamma}$-definable function, in every interval there is an element which is not in the image of the subgroup under these functions.

Corollary 4.3.2 *Assume that τ satisfies Condition 3.1.4. If Γ satisfies (G1)-(G4), then $(\tilde{\mathbb{R}}, \Gamma) \models T \cup dmG_{\Gamma}$.*

Proof: Except for (dmG10), this statement is just the combination of the conditions (G1)-(G4) and Proposition 4.2.2. For (dmG10) note that the image of a countable set under a finite number of $\mathfrak{L}^{\tau}_{\Gamma}$-definable function is countable. Since Γ is countable and every interval is uncountable, there is an element which is not in the image, but in the interval.

$\square$

4.4 Main lemma

First note that the axiom (dmG9) gives us the following Schanuel condition for every model of $T \cup \text{dmG}_{\Gamma}$:

Proposition 4.4.1 *Let $(M, G) \models T \cup dmG$, $g_1, ..., g_m \in G$, $y_1, ..., y_n \in M$ and $\vec{p}$ be a $\mathbb{Q}$-linearly independent tuple over $\mathbb{Q}(\tau)$. If*

$$tr.deg_{\mathbb{Q}(g_1^{\vec{p}}, ..., g_m^{\vec{p}})}(y_1^{\vec{p}}, ..., y_n^{\vec{p}}) < n|\vec{p}| - n,$$

then $g_1^{\vec{p}}, ..., g_m^{\vec{p}}, y_1^{\vec{p}}, ..., y_n^{\vec{p}}$ are multiplicatively dependent.

As noted in the introduction, the following lemma is the crucial bit of the proof. It is a generalization of Lemma 5.12 of [vdDG06]. In contrast to that result, we will have to take care of the fact that the theory T doesn't have quantifier elimination. This is the point where the Schanuel conditions come in.

Lemma 4.4.2 *Let $(M,G) \models T \cup dmG_\Gamma$ and H be a pure subgroup of G containing all interpretations of the constants $\dot\gamma$, where $\gamma \in \Gamma$. Then*

$$\boldsymbol{cl}_T(H) \cap G = H.$$

Proof: The inclusion $H \subset \mathbf{cl}_T(H) \cap G$ is trivial. It is just left to show that whenever $g \in \mathbf{cl}_T(H) \cap G$, then g is also in H. So let $g \in \mathbf{cl}_T(H) \cap G$, $m \in \mathbb{N}$ and $h_1, ..., h_m \in H$ such that $g \in \mathbf{cl}_T(\{h_1, ..., h_m\})$. We can assume that $h_1, ..., h_m$ are $\mathbf{cl}_T$-independent. Further note that since all interpretation of the constants $\dot\gamma$ are in H, for all $\mathbb{Q}$-linearly independent $p_1, ..., p_t \in \mathbb{Q}(\tau)$ all the Mann-Solutions of $G^{[p_1]} \cdot ... \cdot G^{[p_t]}$ are already in $H^{[p_1]} \cdot ... \cdot H^{[p_t]}$ by axiom (dmG8$_{p_1,...,p_t}$).

Case I: There is a $\vec{p} = (p_1, ..., p_t) \in \mathbb{Q}(\tau)^t$ with $(p_1, ..., p_t)$ $\mathbb{Q}$-linearly independent, and

$$tr.deg_{\mathbb{Q}(h_1^{\vec{p}},...,h_m^{\vec{p}})}(g^{\vec{p}}) \leq t - 1. \tag{4.4.1}$$

Suppose that (4.4.1) is witnessed by

$$q(h_1^{\vec{p}}, ..., h_m^{\vec{p}}, g^{\vec{p}}) = 0,$$

where q is a polynomial with coefficients in $\mathbb{Q}$. Further let $q(X_1, ..., X_{mt}, Y_1, ..., Y_t)$ be of the form

$$\sum_{\nu \in \mathbb{N}^t} \sum_{j=1}^{l_\nu} a_{\nu,j} q_{\nu,j}(X_1, ..., X_{mt}) Y^\nu, \tag{4.4.2}$$

where $q_{\nu,j}$ is a monomial in $X_1, ..., X_{mt}$ and $a_{\nu,j} \in \mathbb{Q}$. Set

$$\gamma_{\nu,j} := q_{\nu,j}(h_1^{\vec{p}}, ..., h_m^{\vec{p}}).$$

Since q witnesses (4.4.1), there are two different $\mu_1, \mu_2 \in \mathbb{N}^t$ such that $l_{\mu_i} > 0$ and

$$\sum_{j=1}^{l_{\mu_i}} a_{\mu_i,j} \gamma_{\mu_i,j} \neq 0,$$

for $i = 1, 2$. Let $\sum_{\nu \in \mathbb{N}^t} l_\nu$ be the minimal natural number for which there is a polynomial of the form (4.4.2) witnessing inequality (4.4.1). By definition, $(\gamma_{\nu,j} g^{\nu \cdot \vec{p}})$ is a solution of the equation

$$\sum_\nu \sum_{j=1}^{l_\nu} a_{\nu,j} x_{\nu,j} = 0,$$

where $\nu \cdot \vec{p}$ is the scalar product of $\vec{p}$ and ν. By minimality of $\sum_{\nu\in\mathbb{N}^t} l_\nu$, this solution is non-degenerate. Hence by Proposition 3.3.2 there is $m \in G^{[p_1]} \cdot ... \cdot G^{[p_t]}$ and for every $\nu \in \mathbb{N}^t$ and every $j \in \{1, ..., l_\nu\}$ there is $m_{\nu,j} \in H^{[p_1]} \cdot ... \cdot H^{[p_t]}$ such that

$$\gamma_{\nu,j} g^{\nu\cdot\vec{p}} = m \cdot m_{\nu,j}.$$

For $i = 1, 2$ we get

$$\gamma_{\nu_i,l_{\nu_i}} g^{\nu_i\cdot\vec{p}} = m \cdot m_{\nu_i,l_{\nu_i}}.$$

Hence

$$g^{\nu_1\cdot\vec{p}-\nu_2\cdot\vec{p}} = \frac{m_{\nu_1,l_{\nu_1}}\gamma_{\nu_2,l_{\nu_2}}}{m_{\nu_2,l_{\nu_2}}\gamma_{\nu_1,l_{\nu_1}}} \in H^{[p_1]} \cdot ... \cdot H^{[p_t]}.$$

Thus there are $k_1, ..., k_t \in H$ such that

$$g^{\vec{p}\cdot(\nu_1-\nu_2)} = k_1^{p_1} \cdot ... \cdot k_t^{p_t}.$$

Let $i \in \{1, ..., t\}$ with $\nu_{1,i} \neq \nu_{2,i}$. This implies that

$$\frac{g^{p_i(\nu_{1,i}-\nu_{2,i})}}{k_i^{p_i}} \in G^{[p_i]} \cap G^{[p_1]} \cdot ... \cdot G^{[p_{i-1}]} \cdot G^{[p_{i+1}]} \cdot ... \cdot G^{[p_t]}.$$

By (dmG7) $g^{\nu_{1,i}-\nu_{2,i}} = k_i$. Hence $g^{\nu_{1,i}-\nu_{2,i}} \in H$. Since H is pure in G, $g \in H$.
<u>Case II:</u> Otherwise.
By Proposition 2.3.1, there is $n \in \mathbb{N}$ and $y_1, ..., y_n \in M$ such that $y_1, ..., y_n$ defines g over $h_1, ..., h_m$. Now take n minimal. By Corollary 2.3.4, this implies that there is a $\vec{p} \in \mathbb{Q}(\tau)^{|\vec{p}|}$, whose coordinates are $\mathbb{Q}$-linearly independent, such that

$$tr.deg_{\mathbb{Q}(h_1^{\vec{p}},...,h_m^{\vec{p}})}(g^{\vec{p}}, y_1^{\vec{p}}, ..., y_n^{\vec{p}}) \leq (n + 1)|\vec{p}| - (n + 1). \tag{4.4.3}$$

<u>Case II.1:</u> $tr.deg_{\mathbb{Q}(h_1^{\vec{p}},...,h_m^{\vec{p}},g^{\vec{p}})}(y_1^{\vec{p}}, ..., y_n^{\vec{p}}) < n|\vec{p}| - n$.
By Proposition 4.4.1, $h_1^{\vec{p}}, ..., h_m^{\vec{p}}, g^{\vec{p}}, y_1^{\vec{p}}, ..., y_n^{\vec{p}}$ are multiplicatively dependent. If $g^{\vec{p}}, h_1^{\vec{p}}, ..., h_m^{\vec{p}}$ are multiplicatively dependent, we are back in Case I. Since we have assumed that $h_1, ..., h_m$ are $\mathbf{cl}_T$-independent, $h_1^{\vec{p}}, ..., h_m^{\vec{p}}$ are not multiplicatively dependent. Hence we can assume that there are $k_{i,j}, l_{i,j} \in \mathbb{Z}$ with

$$y_n^r = g^{\sum_{j=1}^{|\vec{p}|} k_{0,j}p_j} \prod_{i=1}^{n-1} y_i^{\sum_{j=1}^{|\vec{p}|} k_{i,j}p_j} \prod_{i=1}^{m} h_i^{\sum_{j=1}^{|\vec{p}|} l_{i,j}p_j}$$

where $r := \sum_{j=1}^{|\vec{p}|} k_{n,j}p_j \neq 0$. Hence

$$y_n = g^{\frac{\sum_{j=1}^{|\vec{p}|} k_{0,j}p_j}{r}} \prod_{i=1}^{n-1} y_i^{\frac{\sum_{j=1}^{|\vec{p}|} k_{i,j}p_j}{r}} \prod_{i=1}^{m} h_i^{\frac{\sum_{j=1}^{|\vec{p}|} l_{i,j}p_j}{r}}.$$

By Theorem 2.3.5, this implies that $y_1, ..., y_{n-1}$ define g over $h_1, ..., h_m$. But this contradicts the minimality of n.
Case II.2: Otherwise, ie.

$$tr.deg_{\mathbb{Q}(h_1^{\vec{p}},...,h_m^{\vec{p}},g^{\vec{p}})}(y_1^{\vec{p}}, ..., y_n^{\vec{p}}) \geq n|\vec{p}| - n.$$

Then by equation (4.4.3)

$$\begin{aligned}(n+1)|\vec{p}| - (n+1) &\geq tr.deg_{\mathbb{Q}(h_1^{\vec{p}},...,h_m^{\vec{p}})}(g^{\vec{p}}, y_1^{\vec{p}}, ..., y_n^{\vec{p}}) \\ &= tr.deg_{\mathbb{Q}(h_1^{\vec{p}},...,h_m^{\vec{p}})}(g^{\vec{p}}) + tr.deg_{\mathbb{Q}(h_1^{\vec{p}},...,h_m^{\vec{p}},g^{\vec{p}})}(y_1^{\vec{p}}, ..., y_n^{\vec{p}}) \\ &\geq tr.deg_{\mathbb{Q}(h_1^{\vec{p}},...,h_m^{\vec{p}})}(g^{\vec{p}}) + n|\vec{p}| - n.\end{aligned}$$

Hence $tr.deg_{\mathbb{Q}(h_1^{\vec{p}},...,h_m^{\vec{p}})}(g^{\vec{p}}) \leq |\vec{p}| - 1$ and we are back in Case I.

□

4.5 Proof of completeness

In this section we will show that $T \cup \mathrm{dmG}_\Gamma$ is complete. Note that the argument in the following proof is the same van den Dries and Günaydin used in [vdDG06]. In previous sections, we have shown the following Main Lemma.

Theorem 4.5.1 *Let $(M, G) \models T \cup dmG_\Gamma$ and H be a pure subgroup of G, which contains all interpretations of the constants $\dot{\gamma}$, where $\gamma \in \Gamma$. Then*

$$\boldsymbol{cl}_T(H) \cap G = H.$$

Further note the following corollaries of the Main Lemma.

Corollary 4.5.2 *Let $(M, G) \models T \cup dmG_\Gamma$ and H be a pure subgroup of G containing all interpretations of the constants $\dot{\gamma}$, where $\gamma \in \Gamma$. If A is $\boldsymbol{cl}_T$-independent over G, then*

$$\boldsymbol{cl}_T(A, H) \cap G = H.$$

Proof: As in the Main Lemma 4.5.1, H is obviously a subset of $\mathbf{cl}_T(A, H) \cap G$. By the Main Lemma 4.5.1 it is only left to show that

$$\mathbf{cl}_T(A, H) \cap G \subseteq \mathbf{cl}_T(H) \cap G. \tag{4.5.1}$$

So let $z \in \mathbf{cl}_T(A, H) \cap G$ and A' a minimal subset of A such that $z \in \mathbf{cl}_T(A', H) \cap G$. For a contradiction, suppose that A' is non-empty and let $a \in A$. By minimality of

A', we have $z \notin \mathbf{cl}_T(A' - \{a\}, H)$. But then the Steinitz Exchange Principle implies that $a \in \mathbf{cl}_T(A' - \{a\}, z, H)$. Since $z \in G$, we get that

$$a \in \mathbf{cl}_T(A' - \{a\}, G).$$

This is a contradiction to the $\mathbf{cl}_T$-independence of A over G. Hence A' is empty and $z \in \mathbf{cl}_T(H) \cap G$. Thus (4.5.1) holds.

□

Corollary 4.5.3 *Let $(M, G) \models T \cup dmG_\Gamma$ and H be a pure subgroup of G containing all interpretations of the constants $\dot{\gamma}$, where $\gamma \in \Gamma$. If A is $\boldsymbol{cl}_T$-independent over G and $g \in G - (\boldsymbol{cl}_T(A, H))$, then*

$$\boldsymbol{cl}_T(A, H, g) \cap G = H_G\langle g\rangle := \{(h \cdot g^k)^{1/m} \mid h \in H, k, m \in \mathbb{N}, hg^k \in G^{[m]}\}.$$

Proof: Since $\mathbf{cl}_T(A, H, g)$ is real closed, $H_G\langle g\rangle \subseteq \mathbf{cl}_T(A, H, g) \cap G$. Because $H_G\langle g\rangle$ is pure in G, Corollary 4.5.2 implies that

$$H_G\langle g\rangle = \mathbf{cl}_T(A, H_G\langle g\rangle) \cap G \supseteq \mathbf{cl}_T(A, H, g) \cap G.$$

□

Let $\mathcal{N} := (M, G(N)), \mathcal{N}' := (M', G(N'))$ be two $(|\Gamma|)^+$-saturated models of $T \cup \mathrm{dmG}_\Gamma$. Then M, M' are models of T. Let $\mathcal{E}$ be the set of all $\mathfrak{L}^\tau_\Gamma$-elementary maps from M to M'. Let $\mathcal{S}$ be the set of all $\beta \in \mathcal{E}$ such that there exist

- a finite subset A of M, and a finite subset A' of M',
- a pure subgroup H of $G(N)$ of cardinality at most $|\Gamma|$ and a pure subgroup H' of $G(N')$ of cardinality at most $|\Gamma|$

such that

1. β is an $\mathfrak{L}^\tau_\Gamma(G)$-isomorphism between $(\mathbf{cl}_T(A, H), H)$ and $(\mathbf{cl}_T(A', H'), H')$,
2. A is $\mathbf{cl}_T$-independent over $G(N)$, and A' is $\mathbf{cl}_T$-independent over $G(N')$ with $\beta(A) = A'$,
3. $\Gamma \leq H, \Gamma \leq H'$.

Note that by Corollary 4.5.2, $(\mathbf{cl}_T(A, H), H)$ resp. $(\mathbf{cl}_T(A', H'), H')$ from the above definition is an $\mathfrak{L}^\tau_\Gamma(G)$-substructure of $(M, G(N))$ resp. $(M', G(N'))$.

Lemma 4.5.4 *$\mathcal{S}$ is a back-and-forth-system of $\mathfrak{L}^{\tau}_{\Gamma}(G)$-isomorphisms.*

Proof: In order to prove this statement, we will show that for every $\beta \in \mathcal{S}$ and every $a \in \mathcal{N}$, there is a $\gamma \in \mathcal{S}$ such that γ extends β and $a \in dom(\gamma)$. In fact, this is enough, because of the symmetry of the setting.
Let $\beta \in \mathcal{S}$ and $a \in \mathcal{N}$. We can assume that $a \notin dom(\beta)$. Further let A, A', H, H' witness that $\beta \in \mathcal{S}$.
Case 1: $a \in G(N)$.
First note that by Proposition 3.5.2, $G(N)$ is regularly dense in M and $G(N')$ is regularly dense in M'. So by Proposition 3.5.3(ii), there is an $a' \in G(N')$ such that for all $h \in H$, $k \in \mathbb{Z}$ and $n > 0$, we have

$$a^k h \in G(N)^{[n]} \text{ iff } a'^k \beta(h) \in G(N')^{[n]}. \tag{4.5.2}$$

In order for a' to have the correct T-type over $\mathbf{cl}_T(A', H')$, let C be the cut of a in $\mathbf{cl}_T(A, H)$. Further let C' be the cut in $\mathbf{cl}_T(A', H')$ corresponding to C under β. Since $\mathcal{N}'$ is saturated, there are $p, q \in M'$ such that all elements in the interval (p, q) realize C'. Since $G(N')$ is dense in M', we can assume that $p, q \in G(N')$. Further since $G(N')$ is regularly dense and $\mathcal{N}'$ is saturated, we can assume that $\bigcap_{n=1}^{\infty} G(N')^{[n]}$ is dense in $G(N')$. Thus take $g \in \bigcap_{n=1}^{\infty} G(N')^{[n]}$ such that $p < ga' < q$. Since ga' realize the cut C' and T is o-minimal, there is an isomorphism $\gamma : \mathbf{cl}_T(A, H, a) \to \mathbf{cl}_{T'}(A', H', ga')$ extending β with $\gamma(a) = ga'$. We set

$$L := \mathbf{cl}_T(A, H, a) \cap G(N) \text{ and } L' := \mathbf{cl}_T(A', H', ga') \cap G(N').$$

By Corollary 4.5.3, we have that $L = H_{G(N)}\langle a \rangle$ and $L' = H'_{G(N')}\langle ga' \rangle$, and that L and L' are pure subgroups of $G(N)$ resp. $G(N')$. Thus by condition (4.5.2), we get that $h \in L$ iff $\gamma(h) \in L'$. Hence γ is an isomorphism between $(\mathbf{cl}_T(A, H, a), L)$ and $(\mathbf{cl}_T(A', H', ga'), L')$.
Case 2: $a \in \mathbf{cl}_T(A, G(N))$.
Let $g_1, ..., g_n \in G(N)$ such that $a \in \mathbf{cl}_{T'}(A, \{g_1, ..., g_n\})$. By using Case 1 n times, we get a $\mu \in \mathcal{S}$ such that $g_1, ..., g_n \in dom(\mu)$ and $A \subseteq dom(\mu)$. Since $dom(\mu)$ is a model of T, we have $a \in dom(\mu)$ with $\mu \in \mathcal{S}$.
Case 3: Otherwise, ie. $a \notin \mathbf{cl}_T(A, G(N))$.
As in Case 1, let C be the cut of a in $\mathbf{cl}_T(A, H)$ and let C' be the corresponding cut of C under β in $\mathbf{cl}_T(A', H')$. Again by saturation, we can assume that there $p, q \in \mathcal{N}'$ such that every element in the interval (p, q) realizes the cut C'. Let $\overline{d}$ be the set A written as a tuple. Let $f_1, ..., f_n$ be 0-definable functions in T. By axiom

(dmG9$_{n,(f_1,...,f_n)}$), we know that there is a $b \in (p,q)$ such that for $i = 1, ..., n$ and every tuple $\overline{g}$ of elements of $G(N')$

$$f_i(\bar{g}, \bar{d}) \neq b.$$

Thus by saturation, there is an $a' \in (p,q)$ such that $a' \notin \mathbf{cl}_T(A', G(N'))$. Since a' realizes the cut C', there is an $\mathfrak{L}^{\tau}_{\Gamma}$-isomorphism γ from $\mathbf{cl}_T(A, a, H)$ to $\mathbf{cl}_T(A', a', H')$ extending β and sending a to a'. Since $a \notin \mathbf{cl}_T(A, G(N))$ and $a' \notin \mathbf{cl}_T(A', G(N'))$, we get that

$$\mathbf{cl}_T(A, a, H) \cap G(N) = H \text{ and } \mathbf{cl}_T(A', a', H') \cap G(N') = H'.$$

Since $\beta(H) = H'$ and γ extends β, we get that γ is an $\mathfrak{L}^{\tau}_{\Gamma}(G)$-isomorphism from $(\mathbf{cl}_T(A, a, H), H)$ to $(\mathbf{cl}_T(A', a', H'), H')$ and $\beta(A \cup \{a\}) = \beta(A' \cup \{a'\})$.

□

Theorem 4.5.5 *$T \cup dmG_{\Gamma}$ is complete.*

Proof: It is enough to show that $\mathcal{S}$ is non-empty. Let $P := \mathbf{cl}_T(\emptyset) \subseteq M$ and $P' := \mathbf{cl}_T(\emptyset) \subseteq M'$. Since $M, M' \models T$, there is an $\mathfrak{L}^{\tau}_{\Gamma}$-isomorphism γ between (P, Γ) and (P', Γ). Note that Γ is pure in $G(N)$ resp. $G(N')$, because of the axiom (dmG5). Hence γ is an $\mathfrak{L}^{\tau}_{\Gamma}(G)$-isomorphism and $\gamma \in \mathcal{S}$.

□

4.6 Proof of the near model completeness

In this section we finally prove Theorem 1.2.2. With this aim in mind, we fix the following notations.

Definition 4.6.1 *(i) Define* the language $\mathfrak{L}^{\tau}_{\Gamma}(G)^+$ *as the language $\mathfrak{L}^{\tau}_{\Gamma}(G)$ together with predicates P_{ϕ} for every $\mathfrak{L}^{\tau}_{\Gamma}(G)$-formula ϕ of the form*

$$\exists \overline{y} \exists \overline{z}\ \psi(\overline{x}, \overline{y}, \overline{z}) \wedge \bigwedge_i G(z_i), \tag{4.6.1}$$

where ψ is a quantifier-free $\mathfrak{L}^{\tau}_{\Gamma}$-formula.
(ii) the $\mathfrak{L}^{\tau}_{\Gamma}(G)^+$-theory $(T \cup \text{dmG}_{\Gamma})^+$ *is the theory $T \cup dmG_{\Gamma}$ together with the following axioms: for every $\mathfrak{L}^{\tau}_{\Gamma}(G)$-formula ϕ of the form* (5.8.1) *add the axiom*

$$P_{\phi}(\overline{x}) \leftrightarrow \phi(\overline{x}).$$

Remember that $\mathfrak{L}_g$ is the language of ordered groups. Let $\mathfrak{L}_{g,\Gamma}$ be $\mathfrak{L}_g$ augmented by a constant symbol $\dot{\gamma}$ for every $\gamma \in \Gamma$.

Definition 4.6.2 *Let φ be a $\mathfrak{L}_{g,\Gamma}$-formula. We say that* the G-restriction of φ, φ_G *is the $\mathfrak{L}_\Gamma^\tau(G)$-formula inductively defined by*

$$\begin{aligned}
&\varphi_G := \varphi && \text{if } \varphi \text{ is atomic,}\\
&\varphi_G := \neg\psi_G && \text{if } \varphi = \neg\psi,\\
&\varphi_G := \chi_G \wedge \psi_G && \text{if } \varphi = \chi \wedge \psi,\\
&\varphi_G := \exists x(G(x) \wedge \psi_G) && \text{if } \varphi = \exists x\psi.
\end{aligned}$$

Proposition 4.6.3 *Let $\varphi(x)$ be an $\mathfrak{L}_{g,\Gamma}$-formula, then φ_G is equivalent to an $\mathfrak{L}_\Gamma^\tau(G)$-formula of the form*

$$\exists \bar{z}(\psi(\bar{x}, \bar{z}) \wedge \bigwedge_i G(z_i)),$$

where $\psi(\bar{x}, z)$ is a quantifier-free $\mathfrak{L}_\Gamma^\tau(G)$-formula.

Proof: By quantifier elimination for the ordered group G (see Theorem 3.5.4), we know that φ is a boolean combinations of formulas for the form

$$\begin{aligned}
&x_1^{k_1} \cdot \ldots \cdot x_n^{k_n} = 1\\
&x_1^{k_1} \cdot \ldots \cdot x_n^{k_n} < 1\\
&x_1^{k_1} \cdot \ldots \cdot x_n^{k_n} \in G^{[d]},
\end{aligned}$$

where $k_i \in \mathbb{Z}$ and $d \in \mathbb{N}_{>0}$. Since $G^{[d]}$ has only finitely many coset in G and each non-trivial coset contains an element γ_i of Γ, the negation of $x_1^{k_1} \cdot \ldots x_n^{k_n} \in G^{[d]}$ is

$$\bigvee_i x_1^{k_1} \cdot \ldots \cdot x_n^{k_n} \in \gamma_i \cdot G^{[d]}.$$

Hence φ_G is of the required form.

□

Below, $\mathcal{S}$ denotes the back-and-forth system between two models $\mathcal{N}, \mathcal{N}'$ of $T \cup \mathrm{dmG}_\Gamma$ constructed in the previous section. For showing Theorem 1.2.2, we have to show

Theorem 4.6.4 $(T \cup dmG_\Gamma)^+$ *has quantifier elimination.*

Proof: In order to show that $(T \cup \mathrm{dmG}_\Gamma)^+$ has quantifier-elimination, let $\vec{a} = (a_1, ..., a_m) \in \mathcal{N}$ and $\vec{b} = (b_1, ..., b_m) \in \mathcal{N}'$ having the same quantifier-free $(\mathfrak{L}^\tau_\Gamma(G))^+$-type. We will now show that then there is an element in the back-and-forth-system $\mathcal{S}$ sending $\vec{a}$ to $\vec{b}$. This implies that $\vec{a}$ and $\vec{b}$ have the same $(\mathfrak{L}^\tau_\Gamma(G))^+$-type. By [H93] Theorem 8.4.1, this implies that $(T \cup \mathrm{dmG}_\Gamma)^+$ has quantifier-elimination.
Let $(a_1, ..., a_r)$ be $\mathbf{cl}_T$-independent over $\mathbf{cl}_T(G(N))$.

(1) $(b_1, ..., b_r)$ is $\mathbf{cl}_T$-independent over $\mathbf{cl}_T(G(N'))$.

Proof: For a contradiction, suppose $(b_1, ..., b_r)$ is $\mathbf{cl}_T$-dependent over $\mathbf{cl}_T(G(N'))$. Then there are $\mathfrak{L}^\tau_\Gamma$-formulas $\varphi(\bar{x}, \bar{y}), \psi(\bar{x}, \bar{y})$ without parameters and there are $g_1, ..., g_n \in G(N')$ such that $\psi(\bar{x}, \bar{y})$ is the formula

$$\varphi(\bar{x}, \bar{y}) \wedge \forall z \ ((\varphi(z, x_2, ..., x_r, \bar{y})) \to (z = x_1))$$

and $\mathcal{N}' \models \psi(b_1, ..., b_r, g_1, ..., g_n)$. By model completeness of T, we can assume that ψ is of the form

$$\exists \bar{u} \chi(\bar{x}, \bar{y}, \bar{u}),$$

where χ is a quantifier-free $\mathfrak{L}^\tau_\Gamma$-formula. Hence

$$\mathcal{N}' \models \exists \bar{y} \exists \bar{u} \chi(b_1, ..., b_r, \bar{y}, \bar{u}) \wedge \bigwedge_{i=1}^{n} G(y_i).$$

Since $\vec{a}$ and $\vec{b}$ have the same quantifier-free $\mathfrak{L}^\tau_\Gamma(G)^+$-type,

$$\mathcal{N} \models \exists \bar{y} \exists \bar{u} \chi(a_1, ..., a_r, \bar{y}, \bar{u}) \wedge \bigwedge_{i=1}^{n} G(y_i).$$

But this contradicts the $\mathbf{cl}_T$-independence of $(a_1, ..., a_r)$ over $\mathbf{cl}_T(G(N))$.

$\square(1)$

By symmetry, we can take r maximal such that, perhaps after rearranging the a_i's, $(a_1, ..., a_r)$ is a maximal such independent subtupel of a and $(b_1, ..., b_r)$ is a maximal such independent subtupel of b. Now let $g_1, ..., g_l \in G(N)$ such that

$$a_{r+1}, ..., a_m \in \mathbf{cl}_T(\{a_1, ..., a_r, g_1, ..., g_l\}).$$

Suppose $\varphi_1(\bar{y}), ..., \varphi_n(\bar{y})$ are $\mathfrak{L}_{g,\Gamma}$-formulas and $\psi_1(\bar{x}, \bar{y}), ..., \psi_t(\bar{x}, \bar{y})$ are $\mathfrak{L}^\tau_\Gamma$-formulas such that

$$\mathcal{N} \models \bigwedge_{i=1}^{n} \varphi_{iG}(g_1, ..., g_l) \wedge \bigwedge_{i=1}^{t} \psi_i(a_1, ..., a_m, g_1, ..., g_l) \wedge \bigwedge_{i=1}^{l} G(g_i).$$

Since T is model complete and because of Proposition 4.6.3, we can assume that the formula

$$\bigwedge_{i=1}^{n} \varphi_{iG}(\bar{y}) \wedge \bigwedge_{i=1}^{t} \psi_i(\bar{x}, \bar{y}) \wedge \bigwedge_{i=1}^{l} G(y_i)$$

is of the form

$$\exists \bar{w} \exists \bar{z} \chi(\bar{w}, \bar{z}, \bar{x}, \bar{y}) \wedge \bigwedge_{i=1}^{l} G(y_i) \wedge \bigwedge_{i=1}^{|\bar{z}|} G(z_i),$$

where χ is quantifier-free $\mathfrak{L}_\Gamma^\tau$-formula. Thus

$$\mathcal{N} \models \exists \bar{y} \exists \bar{w} \exists \bar{z} \chi(\bar{w}, \bar{z}, a_1, ..., a_m, y_1, ..., y_l) \wedge \bigwedge_{i=1}^{l} G(y_i) \wedge \bigwedge_{i=1}^{|\bar{z}|} G(z_i).$$

Since $(a_1, ..., a_m)$ and $(b_1, ..., b_m)$ have the same quantifier-free $(\mathfrak{L}_\Gamma^\tau(G))^+$-type, we have

$$\mathcal{N}' \models \exists \bar{y} \exists \bar{w} \exists \bar{z} \chi(\bar{w}, \bar{z}, b_1, ..., b_m, y_1, ..., y_l) \wedge \bigwedge_{i=1}^{l} G(y_i) \wedge \bigwedge_{i=1}^{|\bar{z}|} G(z_i).$$

Hence by saturation we can get that there are $h_1, ..., h_r \in G(N')$ such that for every $\mathfrak{L}_{g,\Gamma}$-formula $\varphi(\bar{y})$ and every $\mathfrak{L}_\Gamma^\tau$-formula $\psi(\bar{y}, \bar{z})$,

$$\mathcal{N} \models \varphi_G(g_1, ..., g_l) \text{ iff } \mathcal{N}' \models \varphi_G(h_1, ..., h_l)$$
$$\mathcal{N} \models \psi(g_1, ..., g_l, a_1, ..., a_m) \text{ iff } \mathcal{N}' \models \psi(h_1, ..., h_l, b_1, ..., b_m).$$

Hence there is an $\mathfrak{L}_\Gamma^\tau$-elementary map β with

$$\begin{aligned} \beta : \mathbf{cl}_T(\{a_1, ...a_r, g_1, ..., g_l\}) &\to \mathbf{cl}_T(\{b_1, ...b_r, h_1, ..., h_l\}) \\ a_i &\mapsto b_i, \quad \text{for } i = 1, ..., m \\ g_i &\mapsto h_i, \quad \text{for } i = 1, ..., l \end{aligned}$$

and $\beta(\Gamma_{G(N)}\langle g_1, ..., g_l \rangle) = \Gamma_{G(N')}\langle h_1, ..., h_l \rangle$. Since $\{a_1, ..., a_r\}$ is $\mathbf{cl}_T$-independent over $G(N)$, $\{b_1, ..., b_r\}$ is $\mathbf{cl}_T$-independent over $G(N')$ and $\beta(\{a_1, ..., a_r\}) = \{b_1, ..., b_r\}$, we get that $\beta \in \mathcal{S}$.

□

Chapter 5

Groups closed under all definable power functions

5.1 Introduction

In this chapter, we will examine subgroups closed under all definable power functions. Therefore let $\tau \in \mathbb{R} - \mathbb{Q}$. Remember that $\mathbb{Q}(\tau)$ is the field of exponents of $\tilde{\mathbb{R}} := (\mathbb{R}, +, \cdot, 0, 1, x \mapsto \begin{cases} x^\tau, & x > 0, \\ 0, & x \leq 0. \end{cases})$. Now consider a dense multiplicative subgroup Γ of $\mathbb{R}$ with the following properties

(H1) there is a $I \subseteq \overline{\mathbb{Q}}$ such that $\Gamma = I^{[\mathbb{Q}(\tau)]} = \{g_1^{p_1} \cdot \ldots \cdot g_n^{p_n} \mid n \in \mathbb{N}, g_i \in I, p_i \in \mathbb{Q}(\tau)\}$,

(H2) Γ has the Mann property.

We call groups satisfying (H1) *algebraically* $\mathbb{Q}(\tau)$*-generated*. Note that (H1) implies that Γ is divisible, since $\mathbb{Q} \subset \mathbb{Q}(\tau)$. Further (H1) gives us that Γ is countable, since $\mathbb{Q}(\tau)$ is countable. Under Schanuel condition 3.1.2 for τ, $2^{\mathbb{Q}(\tau)}$ satisfies (H1) and (H2) (see Theorem 3.4.5 for a proof). Generally consider the following case.

Proposition 5.1.1 *Assume Condition 3.1.2 holds for τ. Let Γ satisfy (H1) and assume that there is a* finite *set $I \subset \overline{\mathbb{Q}}$ witnessing (H1). Then Γ satisfies (H2).*

Proof: Consider the divisible group generated by $I = (i_1, ..., i_n)$, ie. the group $i_1^{\mathbb{Q}} \cdot \ldots \cdot i_n^{\mathbb{Q}}$. This group has rank at most n. By Theorem 3.3.1, this implies that it has the Mann property. But if the subgroup $i_1^{\mathbb{Q}} \cdot \ldots \cdot i_n^{\mathbb{Q}}$ has the Mann property, Theorem 3.4.5 implies that Γ has the Mann property as well. Hence Γ satisfies (H2).

□

Let Γ be a dense multiplicative subgroup of $\mathbb{R}$ satisfying (H1) and (H2) and let I be the set witnessing (H1). In the first three sections of this chapter we will examine the following Theorem:

Theorem 5.1.2 *Let $(M, G) \models Th(\tilde{\mathbb{R}}, \Gamma)$ and let $\varphi(x_1, ..., x_m)$ be a formula of Schanuel-type. If $a_1, ..., a_m \in M$ and $(M, G) \models \varphi(a_1, ..., a_m)$, then*

$$a_{l_\varphi+1}^{\vec{p_\varphi}}, ..., a_m^{\vec{p_\varphi}} \text{ are multiplicatively dependent over } G.$$

It will be shown that Theorem 5.1.2 holds in the following two cases:

(1) the Conjecture on intersection with tori (see Conjecture 5.3.3) holds, Condition 3.1.2 holds for τ and I is finite,

(2) τ is algebraic and Condition 3.1.4 holds for τ.

The case (1) will be considered in Section 2 and 3, while case (2) will be examined in Section 4. For Γ satisfying (H1) and (H2), we will give a precise first-order axiomatization of $Th(\tilde{\mathbb{R}}, \Gamma)$ in Section 5. Similar to the previous chapter, we will prove the Main Lemma, completeness and near model completeness in the last three sections of this chapter.
Finally note that these results do not follow from the result in Chapter 4. Theorem 6.2.7 in Chapter 6 will show that if (1) or (2) holds for τ, the subgroup $2^{\mathbb{Q}}$ is not definable in $(\tilde{\mathbb{R}}, 2^{\mathbb{Q}(\tau)})$.

5.2 Predimension condition

For rest of this section, let Γ be a group satisfying (H1) and let $I \subset \overline{\mathbb{Q}}$ be a finite set witnessing (H1). By Proposition 5.1.1, Γ satisfies (H2) as well. In the following we define the Predimension condition for Γ and show that under the Schanuel Condition 3.1.2 it holds for Γ.
Given two sets S and F, we will write $mult.dim_F(S)$ for the maximal number of multiplicatively independent elements of S over F. The Predimesion condition states

Condition 5.2.1 *Let Γ be a group satisfying (H1) and let $I \subset \overline{\mathbb{Q}}$ be a finite set witnessing (H1). Let $g_1, ..., g_m \in \Gamma$, $y_1, .., y_n \in \mathbb{R}$, and for $i = 1, ..., n$, let $\vec{p_i} \in \mathbb{Q}(\tau)^{|\vec{p_i}|}$, whose coordinates are $\mathbb{Q}$-linearly independent. Then*

$$tr.deg_{\mathbb{Q}(I,\vec{g})}(y_1^{\vec{p_1}}, ..., y_n^{\vec{p_n}}) - mult.dim_{(I,\vec{g})}(y_1^{\vec{p_1}}, ..., y_n^{\vec{p_n}}) \geq -n.$$

Note that is just axiom (B4) from the introduction.

Theorem 5.2.2 *Assume Condition 3.1.2 holds for τ. Let Γ be a group satisfying (H1) and let $I \subset \overline{\mathbb{Q}}$ be a finite set witnessing (H1). Then the Predimension condition holds.*

Proof: Without loss of generality, we can assume that $I, \vec{g}$ are multiplicatively independent. Since $I, \vec{g}$ are multiplicatively independent, we can assume that there are $h_1, ..., h_l \in I$ and $\vec{q_1}, ..., \vec{q_l}$ $\mathbb{Q}$-linearly independent tuples of $\mathbb{Q}(\tau)$ with $q_{i,1} = 1$ for all $i = 1, .., l$, such that

$$(I, \vec{g}) = (h_1^{\vec{q_1}}, ..., h_l^{\vec{q_l}}).$$

Further, by possibly reducing n and the length of the $\vec{p_i}$, we can also assume that $y_1^{\vec{p_1}}, ..., y_n^{\vec{p_n}}$ are multiplicatively independent over $(h_1^{\vec{q_1}}, ..., h_l^{\vec{q_l}})$. For a contradiction, suppose

$$tr.deg_{\mathbb{Q}(h_1^{\vec{q_1}}, ..., h_l^{\vec{q_l}})}(y_1^{\vec{p_1}}, ..., y_n^{\vec{p_n}}) - mult.dim_{(h_1^{\vec{q_1}}, ..., h_l^{\vec{q_l}})}(y_1^{\vec{p_1}}, ..., y_n^{\vec{p_n}}) < -n.$$

By our assumptions, we know that

$$mult.dim_{(h_1^{\vec{q_1}},...,h_l^{\vec{q_l}})}(y_1^{\vec{p_1}},...,y_n^{\vec{p_n}}) = \sum_{i=1}^{n} |\vec{p_i}|.$$

This implies that

$$tr.deg_{\mathbb{Q}(h_1^{\vec{q_1}},...,h_l^{\vec{q_l}})}(y_1^{\vec{p_1}},...,y_n^{\vec{p_n}}) < \sum_{i=1}^{n} |\vec{p_i}| - n. \tag{5.2.1}$$

By inequality (5.2.1) and the fact that $h_1, ..., h_l$ are algebraic, we have

$$tr.deg_{\mathbb{Q}}(h_1^{\vec{q_1}},...,h_l^{\vec{q_l}}, y_1^{\vec{p_1}},...,y_n^{\vec{p_n}}) < \sum_{i=1}^{l} |\vec{q_i}| + \sum_{i=1}^{n} |\vec{p_i}| - (n+l).$$

Thus $h_1^{\vec{q_1}},...,h_l^{\vec{q_l}}, y_1^{\vec{p_1}},...,y_n^{\vec{p_n}}$ are multiplicatively dependent by Condition 3.1.2. But this contradicts the fact that $h_1^{\vec{q_1}},...,h_l^{\vec{q_l}}$ are multiplicatively independent and $y_1^{\vec{p_1}},...,y_n^{\vec{p_n}}$ are multiplicatively independent over $I, \vec{g}$.

□

5.3 CIT

In this section, it will be shown that under the Conjecture on intersection with tori (CIT), Condition 5.2.1 is first-order expressible. Unfortunately, it is not known whether CIT holds. For corollaries of CIT and its connection to Schanuels Conjecture see [Z02]. Note that τ is arbitrary in this section. In the next section we will consider the special case that τ is algebraic.

Definition 5.3.1 *A* torus $S \subseteq (\mathbb{C}^\times)^n$ *is a set defined by equations of the form*

$$y_1^{m_1} \cdot ... \cdot y_n^{m_n} = 1, \tag{5.3.1}$$

where $m_1, ..., m_n \in \mathbb{Z}$*. Further* S *is said to be* proper *if* $S \neq (\mathbb{C}^\times)^n$*.*

To state the Conjecture on intersection with tori, we further need the concept of an atypical component.

Definition 5.3.2 *Let* $W \subseteq \mathbb{C}^n$ *be an algebraic variety defined over* $\overline{\mathbb{Q}}$*,* $S \subseteq (\mathbb{C}^\times)^n$ *be a torus and* V *be an irreducible component of the variety* $W \cap S$*. We say* V *is an* atypical component of the $W \cap S$ *if*

$$\dim V > \dim W + \dim S - n.$$

Conjecture 5.3.3 *(CIT) Let $W \subseteq \mathbb{C}^n$ be an algebraic variety defined over $\overline{\mathbb{Q}}$. Then there is a finite set $\mathcal{T}(W)$ of proper tori such that for every other torus S and any irreducible component $V \subseteq W \cap S$ with*

$$\dim V > \dim W + \dim S - n,$$

there is an $S' \in \mathcal{T}(W)$ such that $V \subseteq S'$.

The next lemma will show that for elements in the group Γ, equality holds in the Predimension condition.

Lemma 5.3.4 *Assume Condition 3.1.2 holds for τ. Let Γ be a group satisfying (H1) and $I \subset \overline{\mathbb{Q}}$ be a finite set witnessing (H1). Let $g_1, ..., g_m \in \Gamma$, then*

$$tr.deg(I, g_1, ..., g_m) - mult.dim(I, g_1, ..., g_m) = -mult.dim(I).$$

Proof: With out loss of generality, we can assume that $I, g_1, ..., g_m$ are multiplicatively independent. As in the proof of Theorem 5.2.1, we can assume that $I, g_1, ..., g_m$ is of the form $h_1^{\vec{q_1}}, ..., h_l^{\vec{q_l}}$, where $h_i \in I$ and $\vec{q_i} \in \mathbb{Q}(\tau)^{|\vec{q_i}|}$ is $\mathbb{Q}$-linearly independent and $q_{i,1} = 1$. Note that in this case $mult.dim(I) = l$. Further

$$mult.dim(h_1^{\vec{q_1}}, ..., h_l^{\vec{q_l}}) = \sum_{i=1}^{l} |\vec{q_i}|. \tag{5.3.2}$$

Since $I \subset \overline{\mathbb{Q}}$,

$$tr.deg(h_1^{\vec{q_1}}, ..., h_l^{\vec{q_l}}) \leq \sum_{i=1}^{l} |\vec{q_i}| - l. \tag{5.3.3}$$

Thus it is only left to show that in (5.3.3) equality holds. For a contradiction, suppose $tr.deg(h_1^{\vec{q_1}}, ..., h_l^{\vec{q_l}}) < \sum_{i=1}^{l} |\vec{q_i}| - l$. By Condition 3.1.2, this implies that $(h_1^{\vec{q_1}}, ..., h_l^{\vec{q_l}})$ is multiplicatively dependent. This is a contradiction to equation (5.3.2). Hence $tr.deg(h_1^{\vec{q_1}}, ..., h_l^{\vec{q_l}}) = \sum_{i=1}^{l} |\vec{q_i}| - l$ and

$$tr.deg(I, g_1, ..., g_m) - mult.dim(I, g_1, ..., g_m) = \sum_{i=1}^{l} |\vec{q_i}| - l + \sum_{i=1}^{l} |\vec{q_i}| = -l.$$

□

Let $l := mult.dim(I)$ and $h_1, ..., h_l \in \Gamma$ with $l = mult.dim(h_1, ..., h_l)$. Remember for the following that $\mathfrak{Q}_{n,f_1,...,f_p} : \mathbb{R}^n \to \mathbb{R}$ is the function which maps $\vec{a} \in \mathbb{R}^n$ to the sum of the of squares of all determinants of $p \times p$ submatrices of the Jacobian $J_n(f_1, ..., f_p)(\vec{a})$. See Chapter 2 Section 3 for details.

In the following we finally show that the Predimension condition and hence axiom (B4) from the introduction is first-order expressible.

Proposition 5.3.5 *Assume the Conjecture on intersection with tori 5.3.3 holds and Condition 3.1.2 holds for τ. Let Γ be a group satisfying (H1) and $I \subset \overline{\mathbb{Q}}$ be a finite set witnessing (H1). Let $\vec{p}$ be an element of $\mathbb{Q}(\tau)^{|\vec{p}|}$, whose coordinates are $\mathbb{Q}$-linearly independent, and $g_1, ..., g_m \in \Gamma$, $y_1, ..., y_n \in \mathbb{R}$. Further let $W \subseteq \mathbb{R}^{l+m+n\cdot|\vec{p}|}$ be a variety with $\dim W < m + n \cdot |\vec{p}| - n$ and which is defined by polynomials $q_1, ..., q_{n+l+1}$ with coefficients in $\overline{\mathbb{Q}}$ and $q_i = X_i - h_i$ for $i = 1, ..., l$. If $(I, \vec{g}, y_1^{\vec{p}}, ..., y_n^{\vec{p}}) \in W$ and*

$$\mathfrak{Q}_{n\cdot|\vec{p}|,q_{l+1,(I,\vec{g})},\dots,q_{l+n+1,(I,\vec{g})}}(y_1^{\vec{p}}, ..., y_n^{\vec{p}}) \neq 0,$$

then there is a torus $S \in \mathcal{T}(W)$ such that $(I, \vec{g}, y_1^{\vec{p}}, ..., y_n^{\vec{p}}) \in S$.

Proof: By Condition 3.1.2 and Theorem 5.2.2, the following instance of the Predimension condition holds:

$$tr.deg_{\mathbb{Q}(I,g_1,\dots,g_m)}(y_1^{\vec{p}}, ..., y_n^{\vec{p}}) - mult.dim_{(I,g_1,\dots,g_m)}(y_1^{\vec{p}}, ..., y_n^{\vec{p}}) \geq -n.$$

Note that this and Lemma 5.3.4 imply

$$\begin{aligned} tr.deg(I, \vec{g}, y_1^{\vec{p}}, ..., y_n^{\vec{p}}) &= tr.deg(I, \vec{g}) + tr.deg_{\mathbb{Q}(I,\vec{g})}(y_1^{\vec{p}}, ..., y_n^{\vec{p}}) \\ &\geq mult.dim(I, \vec{g}) - l + mult.dim_{(I,\vec{g})}(y_1^{\vec{p}}, ..., y_n^{\vec{p}}) - n \\ &= mult.dim(I, \vec{g}, y_1^{\vec{p}}, ..., y_n^{\vec{p}}) - n - l. \end{aligned}$$

Let W' be the variety defined by polynomial equations over $\mathbb{Q}$ which are satisfied by $(I, \vec{g}, y_1^{\vec{p}}, ..., y_n^{\vec{p}})$. Further let S_0 be the torus defined by all the multiplicative equations of the form (5.3.1) satisfied by $(I, \vec{g}, y_1^{\vec{p}}, ..., y_n^{\vec{p}})$. Note that $\dim W' \geq \dim S_0 - n - l$ by the above. Then

$$\begin{aligned} \dim W + \dim S_0 - m - l - n|\vec{p}| &< m + n \cdot |\vec{p}| - n + \dim S_0 - m - l - n|\vec{p}| \\ &= \dim S_0 - n - l \leq \dim W'. \end{aligned}$$

Thus W' is an atypical component of $W \cap S_0$ and so by Conjecture 5.3.3 there is a torus $S \in \mathcal{T}(W)$ such that

$$(I, \vec{g}, y_1^{\vec{p}}, ..., y_n^{\vec{p}}) \in S.$$

□

The definition of a formula of Schanuel-type and Proposition 5.3.5 directly imply the following corollary. For given $\vec{p} \in \mathbb{Q}(\tau)^n$, we write $I^{\vec{p}}$ for the tuple $(i_1^{\vec{p}}, ..., i_l^{\vec{p}})$.

Corollary 5.3.6 *Assume that the Conjecture on intersection with tori 5.3.3 holds and Condition 3.1.2 holds for τ. Let Γ be a group satisfying (H1), $I \subset \overline{\mathbb{Q}}$ be a finite set witnessing (H1) and let $\varphi(x_1, ..., x_{m+l})$ be a formula of Schanuel-type. Then there is a finite set $\mathcal{T}(\varphi)$ of proper tori such that for all $a_1, ..., a_m \in \mathbb{R}$ with $(\tilde{\mathbb{R}}, \Gamma) \models \varphi(I, a_1, ..., a_m)$ there is $S \in \mathcal{T}(\varphi)$ with $(I^{\vec{p^\varphi}}, a_1^{\vec{p^\varphi}}, ..., a_m^{\vec{p^\varphi}}) \in S$.*

Proof: This corollary directly follows from Proposition 5.3.5. Let n be the natural number with $|I| = n$. Further let $I = \{i_1, ..., i_n\}$ and let r_j be the polynomial $X_j - i_j$. Then the set $\mathcal{T}(\varphi)$ is just the set $\mathcal{T}(W)$, where W is the variety given by the polynomials $r_1, ..., r_n$ and $q_1^\varphi, ..., q_{m-l_\varphi+1}^\varphi$.

□

In particular, note that because of the finiteness of $\mathcal{T}(\varphi)$, Corollary 5.3.6 is first-order expressible. This allows us to add it as an axiom to the axiomatization of the theory of $(\tilde{\mathbb{R}}, \Gamma)$.

5.4 Group Schanuel condition for algebraic τ

In this section we consider the special case, where τ is algebraic of degree d. Further let Γ be a dense multiplicative subgroup of $\mathbb{R}$ satisfying (H1) and (H2). Note that in this section, we do *not* assume that (H1) is witnessed by a finite set I.

We will show that in this case, the Uniform Schanuel Condition 3.1.4 implies the following Theorem which is a version of the Predimension condition 5.2.1. Note that the conclusion of the Theorem is first-order expressible. More generally, note that we only know that Condition 3.1.4 holds for generic τ, but we do not know anything about the case when τ is algebraic.

Theorem 5.4.1 *Assume that τ is algebraic of degree d and satisfies Condition 3.1.4. Let Γ be group satisfying (H1) and (H2). Let $\varphi(x_1, ..., x_m)$ be a formula of Schanuel-type. Then there is a natural number $L(\varphi)$ such that for $a_1, ..., a_m \in \mathbb{R}$, if $(\tilde{\mathbb{R}}, \Gamma) \models \varphi(a_1, ..., a_m)$, then there are $m_{i,j} \in \mathbb{Z}$, not all zero and $|m_{i,j}| \leq L(\varphi)$ such that*

$$\prod_{i=l_\varphi+1}^{m} \prod_{j=1}^{|\vec{p^\varphi}|} a_i^{m_{i,j} \cdot p_j^\varphi} \in \Gamma. \tag{5.4.1}$$

In order to prove this statement, we need two lemmata. We fix the following notation. Let $I \subseteq \overline{\mathbb{Q}}$ be the set that witnesses (H1) for Γ. We define a group Δ by

$$\Delta := \{i_1^{q_1} \cdot ... \cdot i_n^{q_n} \mid n \in \mathbb{N}, i_1, ..., i_n \in I, q_1, ..., q_n \in \mathbb{Q}\}$$

Because Γ is divisible, Δ is a subgroup of Γ. Further note that Δ is a subset of $\overline{\mathbb{Q}}$.

Lemma 5.4.2 *Γ is definable in $(\tilde{\mathbb{R}}, \Delta)$.*

Proof: Since τ is algebraic of degree d, $\Gamma = \Delta \cdot \Delta^{[\tau]} \cdot ... \cdot \Delta^{[\tau^{d-1}]}$. This is definable in $(\tilde{\mathbb{R}}, \Delta)$ by

$$\psi(y) := \exists x_0 ... \exists x_{d-1} \bigwedge_{i=0}^{d-1} G(x_i) \wedge y = x_0 \cdot x_1^{\tau} \cdot ... \cdot x_{d-1}^{\tau^{d-1}}.$$

□

Lemma 5.4.3 *Assume τ satisfies Condition 3.1.4. Let $t, m \in \mathbb{N}$ with $t < m$, let $\vec{p_i} \in \mathbb{Q}(\tau)^{|\vec{p_i}|}$ be a $\mathbb{Q}$-linearly independent tuple, for $i = 1, ..., m$, and let $q_i \in \mathbb{Q}[X_1, ..., X_{\sum_{i=1}^m |\vec{p}|}]$, for $i = 1, ..., m - t + 1$. Then there is a natural number $N = N(t, q_1, ..., q_{m-t+1}, \vec{p_1}, ..., \vec{p_m})$ such that for every $a_1, ..., a_t \in \Delta$ and $a_{t+1}, ..., a_m \in \mathbb{R}$ with*

$$q_i(a_1^{\vec{p_1}}, ... a_t^{\vec{p_t}}, a_{t+1}^{\vec{p_{t+1}}}, ..., a_m^{\vec{p_m}}) = 0, \text{ for all } i = 1, ..., m - t + 1,$$

and

$$\mathfrak{Q}_{\sum_{i=t+1}^m |\vec{p_i}|, q_{1,\vec{g}}, ..., q_{m-t+1,\vec{g}}}(a_{t+1}^{\vec{p_{t+1}}}, ..., a_m^{\vec{p_m}}) \neq 0,$$

where

$$\vec{g} := (a_1^{\vec{p_1}}, ..., a_t^{\vec{p_t}}),$$

there are $m_{i,j} \in \mathbb{Z}$, not all zero and $|m_{i,j}| \leq N$ such that

$$\prod_{i=1}^{m} \prod_{j=1}^{|\vec{p_i}|} a_i^{m_{i,j} \cdot p_{i,j}} = 1. \tag{5.4.2}$$

Proof: First note that Δ is a subset of $\overline{\mathbb{Q}}$ and hence it satisfies Condition (G1). The proof of Proposition 4.2.2 can be easily adapted to this case. The only difference between the two settings is that now $\vec{p_i}$ is not necessarily equal to $\vec{p_j}$, for $i \neq j$. But this difference was not used in the proof of Proposition 4.2.2.

□

By the uniformity in Lemma 5.4.3, we get that the result also holds in arbitrary models of the theory of $(\tilde{\mathbb{R}}, \Delta)$.

Corollary 5.4.4 *Assume τ satisfies Condition 3.1.4. Let $m \in \mathbb{N}$ and let $\vec{p_i} \in \mathbb{Q}(\tau)^{|\vec{p_i}|}$ be a $\mathbb{Q}$-linearly independent tuple, for $i = 1, ..., m$. Further let $(M, G) \models Th((\tilde{\mathbb{R}}, \Delta))$ and $a_1, ..., a_m \in M$. If there is $l < m$ such that $a_i \in G$ for all $i = 1, ..., l$ and*

$$tr.deg_{\mathbb{Q}(a_1^{\vec{p_1}},...,a_l^{\vec{p_l}})}(a_{l+1}^{\vec{p_{l+1}}}, ..., a_m^{\vec{p_m}}) < \sum_{i=l+1}^{m} |\vec{p_i}| - (m - l + 1),$$

then $a_1^{\vec{p_1}}, ..., , a_m^{\vec{p_m}}$ are multiplicatively dependent.

Proof of Theorem 5.4.1: By Lemma 5.4.2, Γ is definable in $(\tilde{\mathbb{R}}, \Delta)$. Suppose $(M, G) \models Th((\tilde{\mathbb{R}}, \Delta))$ and let $H := G \cdot G^{[\tau]} \cdot ... \cdot G^{[\tau^{d-1}]}$. By the definability of H and Γ, this implies that $(M, G, H) \models Th((\tilde{\mathbb{R}}, \Delta, \Gamma)$. Let $a_1, ..., a_m \in M$ such that $(M, H) \models \varphi(a_1, ..., a_m)$. For readability set $\vec{q} := \vec{p_\varphi}$ and set $v := l_\varphi$.

(1) $a_{l+1}^{\vec{q}}, ..., a_m^{\vec{q}}$ are multiplicatively dependent over H.

Proof: First note that $(M, H) \models \varphi(a_1, ..., a_m)$ implies that $a_1, ..., a_v \in H$ and

$$tr.deg_{\mathbb{Q}(a_1^{\vec{q}},...,a_v^{\vec{q}})}(a_{v+1}^{\vec{q}}, ..., a_m^{\vec{q}}) < (m - v + 1)|\vec{q}| - (m - (v + 1)).$$

By the definition of H, for every $i = 1, ..., v$ there are $b_{i,0}, ..., b_{i,d-1} \in G$ such that

$$a_i = b_{i,0} \cdot b_{i,1}^{\tau} \cdot ... \cdot b_{i,d-1}^{\tau^{d-1}}.$$

Let $c_1, ..., c_k \in G$ be a maximal multiplicatively independent subset of

$$\{b_{i,j} \mid i = 1, ..., v; j = 0, ..., d - 1\}$$

and let $t_1, ..., t_s$ be a basis of the $\mathbb{Q}$-vector space generated by

$$1, \tau, ..., \tau^{d-1}, q_1, ..., q_{|\vec{q}|} \text{ with } t_1 = 1.$$

Set $\vec{t} := (t_1, ..., t_s) \in \mathbb{Q}(\tau)^s$. Then

$$tr.deg_{\mathbb{Q}(c_1^{\vec{t}},...,c_k^{\vec{t}})}(a_{v+1}^{\vec{q}}, ..., a_m^{\vec{q}}) < (m - l + 1)|\vec{q}| - (m - (l + 1)).$$

For every $i = 1, ..., k$ we can now take a subvectors $\vec{r_i}$ of $\vec{t}$ such that $r_{i,1} = 1$, $c_1^{\vec{r_1}}, ..., c_k^{\vec{r_k}}$ are multiplicatively independent and

$$tr.deg_{\mathbb{Q}(c_1^{\vec{r_1}},...,c_k^{\vec{r_k}})}(a_{v+1}^{\vec{q}}, ..., a_m^{\vec{q}}) < (m - v + 1)|\vec{q}| - (m - (v + 1)).$$

By Corollary 5.4.4, we know that $c_1^{\vec{r_1}}, ..., c_k^{\vec{r_k}}, a_{v+1}^{\vec{q}}, ..., a_m^{\vec{q}}$ are multiplicately dependent. Since $c_1^{\vec{r_1}}, ..., c_k^{\vec{r_k}}$ are multiplicatively independent, $a_{v+1}^{\vec{q}}, ..., a_m^{\vec{q}}$ must be multiplicatively dependent over $c_1^{\vec{r_1}}, ..., c_k^{\vec{r_k}}$. But $c_i^{r_{i,j}} \in H$ for every i, j, and hence $a_{v+1}^{\vec{q}}, ..., a_m^{\vec{q}}$ are multiplicatively dependent over H.

$\square(1)$

The statement now follows from compactness: let $\mathfrak{L}_\mathfrak{c}$ be the language of $(\tilde{\mathbb{R}}, \Delta, \Gamma)$ augmented by constant symbols $c_1, ..., c_n$. Let T_c be the theory of $(\tilde{\mathbb{R}}, \Delta, \Gamma)$ together with the following axioms:

(T_c1) $\varphi(c_1, ..., c_n)$,

(T_c2) $\prod_{i=t+1}^{n} \prod_{j=1}^{|\vec{q}|} c_i^{m_{i,j} \cdot q_j} \notin \Gamma$, for $m_{i,j} \in \mathbb{Z}$ not all zero.

By (1) T_c is inconsistent. Hence by compactess, there is a natural number L such that the theory of $(\tilde{\mathbb{R}}, \Delta, \Gamma)$ together with the axiom (T_c1) and

(T_c2_L) $\prod_{i=t+1}^{n} \prod_{j=1}^{|\vec{q}|} c_i^{m_{i,j} \cdot q_j} \notin \Gamma$, for $m_{i,j} \in \mathbb{Z}$ not all zero and $|m_{i,j}| \leq L$,

is inconsistent. Hence this natural number L has the property that for all $a_1, ..., a_n \in \mathbb{R}$ with $(\tilde{\mathbb{R}}, \Gamma) \models \varphi(a_1, ..., a_n)$, there are $m_{i,j} \in \mathbb{Z}$, not all zero and $|m_{i,j}| \leq L$ such that

$$\prod_{i=t+1}^{m} \prod_{j=1}^{|\vec{q}|} a_i^{m_{i,j} \cdot q_j} \in \Gamma$$

holds.

$\square$

5.5 Axioms KdmG

For this section, let Γ be a dense multiplicative subgroup of $\mathbb{R}$ satisfying (H1) and (H2). We now can define the first order axioms of algebraically $\mathbb{Q}(\tau)$-generated dense multiplicative subgroups $KdmG$ in the language $\mathfrak{L}_\Gamma^\tau(G)$. In the definition of $KdmG$ we will use the following abbreviation. For all $m \in \mathbb{N}$ and $\vec{\alpha} \in \mathbb{Q}^m$, let $MS(\vec{\alpha})$ be the finite set of non-degenerate solutions in Γ of the equation

$$\alpha_1 x_1 + ... + \alpha_m x_m = 1.$$

Definition 5.5.1 *We define the $\mathfrak{L}_\Gamma^\tau(G)$-theory $KdmG$ by the following set of axioms:*

(KdmG1) $\forall x > 0 \forall y > 0((x < y) \rightarrow (\exists z(G(z) \wedge x < z < y)))$,

(KdmG2) $\forall x \forall y((G(x) \wedge G(y) \rightarrow G(x \cdot y))$,

(KdmG3) $\forall x \forall y((G(x) \wedge x \cdot y = 1) \rightarrow G(y))$,

(KdmG4$_{\gamma,\delta}$) $G(\dot{\gamma})$, $\dot{\gamma}\dot{\delta} = (\dot{\delta\gamma})$, $\dot{1} = 1, \dot{\gamma}(\dot{\gamma^{-1}}) = 1$, *for all* $\gamma, \delta \in \Gamma$,

(KdmG5$_p$) $\forall x(G(x) \to G(x^p))$, *for all* $p \in \mathbb{Q}(\tau)$,

(KdmG6$_{m,\vec{\alpha}}$) $\forall x_1 ... \forall x_m \big(\big(\bigwedge_{i=1}^m G(x_i) \wedge \sum_{i=1}^m \alpha_i x_i = 1 \wedge \bigwedge_{D \subset 2^m} \sum_{i \in D} \alpha_i x_i \neq 0 \big)$ $\to (\bigvee_{(\gamma_1,...,\gamma_m) \in MS(\vec{p},\vec{\alpha})} \bigwedge_{i=1}^m x_i = \gamma_i))$, *for all* $m \in \mathbb{N}$ *and* $\vec{\alpha} \in \mathbb{Q}^m$,

(KdmG7$_\varphi$) $\varphi(x_1, ..., x_m) \to (\bigvee_{k_{l_\varphi+1,1}=1}^{L(\varphi)} ... \bigvee_{k_{m,|\vec{p}^\varphi|}=1}^{L(\varphi)} G(\prod_{i=l_\varphi}^m \prod_{j=1}^{|\vec{p}^\varphi|} x_i^{k_j \cdot p_j^\varphi}))$, *for all* $\mathfrak{L}_\Gamma^\tau(G)$-*formulas* φ *of Schanuel-type over* G,

(KdmG8$_{d,\overline{f}}$) $\forall \overline{y} \forall a \forall b \exists z \ a < z < b \wedge (\forall \overline{x}(G(\overline{x}) \to (\bigwedge_{i=1}^d f_i(\overline{x}, \overline{y}) \neq z)))$, *for all* $d \in \mathbb{N}$, *for all* $i \leq d$ *let* $f_i(\overline{x}, \overline{y})$ *be a* $\mathfrak{L}_\Gamma^\tau$*-0-definable function.*

Note that (KdmG1), (KdmG2), (KdmG3), (KdmG4) and (KdmG5) force the interpretation of G to be a dense multiplicative subgroup of $\mathbb{R}$ with subgroup Γ and to be closed under all power function definable in $\tilde{\mathbb{R}}$. The axiom (KdmG6$_d$) states that all Mann solutions of G are actually in the subgroup Γ. Axiom (KdmG7$_\varphi$) forces solutions of formulas of Schanuel-type to be multiplicatively dependent over the group G. Finally axiom (KdmG8) is the statement that for a finite number of $\mathfrak{L}^\tau$-definable functions, in every interval there is an element which is not in the image of the subgroup G under these functions.

Corollary 5.5.2 *Assume that* τ *is algebraic and satisfies Condition 3.1.4. Let* Γ *be a group satisfying (H1) and (H2). Then* $(\tilde{\mathbb{R}}, \Gamma) \models T \cup KdmG$.

Proof: Except for (KdmG8), this statement follows from (H1)-(H2) and Theorem 5.4.1. For (KdmG8) note that the image of a countable set under a finite number of $\mathfrak{L}_\Gamma^\tau$-definable function is countable. Since Γ is countable and every interval is uncountable, there is an element which is not in the image, but in the interval.

□

Further consider the case that (H1) is witnessed by finite set $I \subset \overline{\mathbb{Q}}$. Consider the following sentences:

(KmdG7') $\varphi(I, x_1, ..., x_n) \to \bigvee_{S \in \mathcal{T}(\varphi)} (I^{p^{\vec{\varphi}}}, x_1^{p^{\vec{\varphi}}}, ..., x_n^{p^{\vec{\varphi}}}) \in S$, for all $\mathfrak{L}_\Gamma^\tau(G)$-formulas φ of Schanuel-type over G.

Let $KmdG'$ be theory $KmdG$ after replacing $(KdmG7)$ by $(KmdG7')$.

Corollary 5.5.3 *Assume that Conjecture 5.3.3 holds and that Condition 3.1.2 holds for* τ*. Let* Γ *be a group which satisfies (H1), and let* $I \subset \overline{\mathbb{Q}}$ *be a finite set witnessing (H1) for* Γ*. Then* $(\tilde{\mathbb{R}}, \Gamma) \models T \cup KdmG'$*.*

Proof: The statement directly follows from Corollary 5.3.6 and the proof of Corollary 5.5.2.

□

5.6 Main lemma

In this section, we will show that the Main Lemma holds for $T \cup$ KdmG. First note that by $(KdmG7)$, we have:

Theorem 5.6.1 *Let* $(M, G) \models T \cup KdmG$ *and let* $\varphi(x_1, ..., x_m)$ *be a formula of Schanuel-type. If* $a_1, ..., a_m \in M$ *and* $(M, G) \models \varphi(a_1, ..., a_m)$*, then*

$$a_{l_\varphi+1}^{\vec{p^\varphi}}, ..., a_m^{\vec{p^\varphi}} \text{ are multiplicatively dependent over } G.$$

In fact, in the case that (H1) for Γ is witnessed by a finite set I, the same holds for any model of $T \cup KdmG'$.

Theorem 5.6.2 *Let* $(M, G) \models T \cup KdmG'$ *and let* $\varphi(x_1, ..., x_m)$ *be a formula of Schanuel-type. If* $a_1, ..., a_m \in M$ *and* $(M, G) \models \varphi(a_1, ..., a_m)$*, then*

$$a_{l_\varphi+1}^{\vec{p^\varphi}}, ..., a_m^{\vec{p^\varphi}} \text{ are multiplicatively dependent over } G.$$

Proof: Let $I = \{i_1, ..., i_{|I|}\}$. Let $\psi(y_1, ..., y_{|I|}, x_1, ..., x_m)$ be the formula defined by

$$\varphi(x_1, ..., x_m) \wedge \bigwedge_{i=1}^{|I|} G(y_i).$$

It directly follows from φ being a formula of Schanuel-type that ψ is a formula of Schanuel-type as well. Further note that $(M, G) \models \psi(I, a_1, ..., a_m)$. By maybe decreasing l_ψ and changing the q_i^ψ, we can assume that

$$i_1^{\vec{p^\varphi}}, ..., i_{|I|}^{\vec{p^\varphi}}, a_1^{\vec{p^\varphi}}, ..., a_{l_\psi}^{\vec{p^\varphi}} \text{ are multiplicatively independent.} \tag{5.6.1}$$

Note by $(KdmG5)$, these are all elements of G. Further by $(KmdG7')$ this implies that

$$i_1^{\vec{p^\varphi}}, ..., i_{|I|}^{\vec{p^\varphi}}, a_1^{\vec{p^\varphi}}, ..., a_m^{\vec{p^\varphi}}$$

are multiplicatively dependent and hence by (5.6.1) $a_{l_\varphi+1}^{\vec{p^\varphi}}, ..., a_m^{\vec{p^\varphi}}$ are multiplicatively dependent over G.

□

In the following we will use $\tilde{T}$ for either $T \cup KdmG$ or $T \cup KdmG'$. If $\tilde{T}$ is $T \cup KdmG'$, we assume from now on that a finite set I witness that Γ satisfies (H1).
In order to show the Main Lemma, the following proposition shows that the axiom (KdmG6) implies a general Schanuel-style condition for models of $\tilde{T}$.

Proposition 5.6.3 *Let $(M,G) \models \tilde{T}$, $h_1,...,h_n \in G$ and $\vec{p} \in \mathbb{Q}(\tau)^{|\vec{p}|}$. If $g \in G$ with*

$$tr.deg_{\mathbb{Q}(h_1,...,h_n)}(g^{\vec{p}}) < |\vec{p}|,$$

then there is $\gamma \in \Gamma$ such that $g^{\vec{p}}, h_1, ..., h_n, \gamma$ are multiplicatively dependent.

<u>Proof:</u> Set $m := |\vec{p}|$ and suppose

$$q(h_1,...,h_n,g^{\vec{p}}) = 0,$$

where q is a polynomial with coefficients in $\mathbb{Q}$ witnessing the algebraic dependency of $g^{\vec{p}}$ over $h_1,...,h_n$. Further let $q(X_1,...,X_n,Y_1,...,Y_m)$ be of the form

$$\sum_{\nu \in \mathbb{N}^m} \sum_{j=1}^{l_\nu} a_{\nu,j} q_{\nu,j}(X_1,...,X_n) Y^\nu, \tag{5.6.2}$$

where $q_{\nu,j}$ is a monomial in $X_1,...,X_n$ and $a_{\nu,j} \in \mathbb{Q}$. Set

$$\gamma_{\nu,j} := q_{\nu,j}(h_1,...,h_n).$$

We can assume that there are two different $\mu_1, \mu_2 \in \mathbb{N}^m$ such that $l_{\mu_i} > 0$ and

$$\sum_{j=1}^{l_{\mu_i}} a_{\mu_i,j} \gamma_{\mu_i,j} \neq 0,$$

for $i = 1,2$. Let $\sum_{\nu \in \mathbb{N}^m} l_\nu$ be the minimal natural number for which there is a polynomial of the form (5.6.2) witnessing the algebraic dependency of $g^{\vec{p}}$ over $h_1,...,h_n$. By definition $(\gamma_{\nu,j} g^{\nu \cdot \vec{p}})$ is a solution of

$$\sum_{\nu} \sum_{j=1}^{l_\nu} a_{\nu,j} x_{\nu,j} = 0,$$

where $\nu \cdot \vec{p}$ is the scalar product of $\vec{p}$ and ν. By minimality of $\sum_{\nu \in \mathbb{N}^m} l_\nu$, this solution is non-degenerate. Note that by (KdmG6) G has the Mann property and further all

Mann solutions of G are in Γ. Hence by Proposition 3.3.2 there is $m \in G$ such that for every $\nu \in \mathbb{N}^m$ and every $j \in \{1, ..., l_\nu\}$ there is $m_{\nu,j} \in \Gamma$ with

$$\gamma_{\nu,j} g^{\nu \cdot \vec{p}} = m \cdot m_{\nu,j}.$$

Since $l_{\mu_1} > 0$, $l_{\mu_2} > 0$, we get

$$\gamma_{\mu_1,l_{\mu_1}} g^{\mu_1 \cdot \vec{p}} = m \cdot m_{\mu_1,l_{\mu_1}}.$$

and

$$\gamma_{\mu_2,l_{\mu_2}} g^{\mu_2 \cdot \vec{p}} = m \cdot m_{\mu_2,l_{\mu_2}}.$$

Hence

$$\frac{m_{\mu_1,l_{\mu_1}} \gamma_{\mu_1,l_{\mu_1}}}{m_{\mu_2,l_{\mu_2}} \gamma_{\mu_2,l_{\mu_2}}} g^{(\mu_1 - \mu_2)\cdot \vec{p}} = 1 \tag{5.6.3}$$

Set $\gamma := m_{\mu_1,l_{\mu_1}} \cdot (m_{\mu_2,l_{\mu_2}})^{-1} \in \Gamma$. Then by equation (5.6.3), $g^{\vec{p}}, h_1, ..., h_n, \gamma$ are multiplicatively dependent.

□

After establishing this Schanuel-style condition for models of $\tilde{T}$, we are now ready to prove the Main Lemma.

Lemma 5.6.4 *Let $(M, G) \models \tilde{T}$ and H be a subgroup of G, which is closed under all definable power function in M and contains all interpretations of the constants $\dot{\gamma}$, where $\gamma \in \Gamma$. Then*

$$\boldsymbol{cl}_T(H) \cap G = H.$$

Proof: The inclusion $H \subset \mathbf{cl}_T(H) \cap G$ is trivial. It is just left to show that whenever $g \in \mathbf{cl}_T(H) \cap G$, then g is also in H. So let $g \in \mathbf{cl}_T(H) \cap G$, $m \in \mathbb{N}$ and $h_1, ..., h_m \in H$ such that $g \in \mathbf{cl}_T(\{h_1, ..., h_m\})$.
By Proposition 2.3.1, there is $n \in \mathbb{N}$, and $y_1, ..., y_n \in M$ such that $y_1, ..., y_n$ define g over $h_1, ..., h_m$. Now take m and $h_1, ..., h_m$ such that n is minimal, ie. such that there are *no* $s \in \mathbb{N}$, $h_1, ..., h_s \in H$, $l \in \mathbb{N}$ and $y_1, ..., y_l \in M$ satisfying

(1) $l < n$ and

(2) $y_1, ..., y_l$ define g over $h_1, ..., h_s$.

By Corollary 2.3.4, we know that there is a $\mathbb{Q}$-linearly independent $\vec{p} \in \mathbb{Q}(\tau)^{|\vec{p}|}$ such that

$$tr.deg_{\mathbb{Q}(h_1^{\vec{p}},...,h_m^{\vec{p}})}(g^{\vec{p}}, y_1^{\vec{p}}, ..., y_n^{\vec{p}}) \leq (n+1)|\vec{p}| - (n+1). \tag{5.6.4}$$

We can further assume that $p_1 = 1$.

<u>Case I:</u> $tr.deg_{\mathbb{Q}(h_1^{\vec{p}},...,h_m^{\vec{p}},g^{\vec{p}})}(y_1^{\vec{p}}, ..., y_n^{\vec{p}}) < n|\vec{p}| - n$.

We will now show that this contradicts the minimality of n. Therefore we have to consider the following to cases. With out loss of generalization we can assume that there is $l \in \mathbb{N}$ with $l < n$ such that $y_1, ..., y_l \in G$ and $y_{l+1}^{\vec{p}}, ..., y_n^{\vec{p}}$ are multiplicatively independent over G. Since G is closed under all power function, this implies that $y_i^{p_j} \in G$ for all $i = 1, ..., l$ and $j = 1, ..., |\vec{p}|$.

<u>Case I.a:</u> $tr.deg_{\mathbb{Q}(h_1^{\vec{p}},...,h_m^{\vec{p}},g^{\vec{p}},y_1^{\vec{p}},...,y_l^{\vec{p}})}(y_{l+1}^{\vec{p}}, ..., y_n^{\vec{p}}) < (n-l)|\vec{p}| - (n-l)$.

By Theorem 5.6.1 resp. Theorem 5.6.2 and our assumption for Case I.a imply that $y_{l+1}, ..., y_n$ are multiplicatively dependent over G. This is a contradiction to our assumption about $y_1, ..., y_n$.

<u>Case I.b:</u> Otherwise, ie.

$$tr.deg_{\mathbb{Q}(h_1^{\vec{p}},...,h_m^{\vec{p}},g^{\vec{p}},y_1^{\vec{p}},...,y_l^{\vec{p}})}(y_{l+1}^{\vec{p}}, ..., y_n^{\vec{p}}) \geq (n-l)|\vec{p}| - (n-l).$$

This implies together with the assumptions for Case I that

$$\begin{aligned}
&tr.deg_{\mathbb{Q}(h_1^{\vec{p}},...,h_m^{\vec{p}},g^{\vec{p}})}(y_1^{\vec{p}}, ..., y_l^{\vec{p}}) \\
&\qquad = tr.deg_{\mathbb{Q}(h_1^{\vec{p}},...,h_m^{\vec{p}},g^{\vec{p}})}(y_1^{\vec{p}}, ..., y_n^{\vec{p}}) - tr.deg_{\mathbb{Q}(h_1^{\vec{p}},...,h_m^{\vec{p}},g^{\vec{p}},y_1^{\vec{p}},...,y_l^{\vec{p}})}(y_{l+1}^{\vec{p}}, ..., y_n^{\vec{p}}) \\
&\qquad < n|\vec{p}| - n - ((n-l)|\vec{p}| - (n-l)) \\
&\qquad = l|\vec{p}| - l.
\end{aligned}$$

So we can assume that

$$tr.deg_{\mathbb{Q}(h_1^{\vec{p}},...,h_m^{\vec{p}},g^{\vec{p}},y_1^{\vec{p}},...,y_{l-1}^{\vec{p}})}(y_l^{\vec{p}}) < |\vec{p}|.$$

Because $y_1, ..., y_l \in G$, Proposition 5.6.3 implies that

$$y_l^{\sum_{j=1}^{|\vec{p}|} k_{l,j}p_j} = \gamma \cdot \prod_{i=1}^{m} \prod_{j=1}^{|\vec{p}|} h_i^{l_{i,j}p_j} \cdot \prod_{i=0}^{l-1} \prod_{j=1}^{|\vec{p}|} y_i^{k_{i,j}p_j}, \tag{5.6.5}$$

where $k_{i,j}, l_{i,j} \in \mathbb{Z}$, $\gamma \in \Gamma$, $y_0 := g$ and $\sum_{j=1}^{|\vec{p}|} k_{l,j}p_{l,j} \neq 0$. Trivially, since $y_1, ..., y_n$ define g over $h_1, ..., h_m$, we also know that $y_1, ...y_n$ define g over $\gamma, h_1, ..., h_m$. Note further that equation (5.6.5) implies that

$$y_l = \gamma^{t_0} \cdot \prod_{i=1}^{m} h_i^{t_i} \cdot \prod_{i=0}^{l-1} y_i^{s_i},$$

where $s_i, t_i \in \mathbb{Q}(\tau)$. Hence we get by Corollary 2.3.5 that already $y_1, ..., y_{l-1}, y_{l+1}, ..., y_n$ define g over $\gamma, h_1, ..., h_m$. This contradicts the minimality of n.
Case II: Otherwise, ie.

$$tr.deg_{\mathbb{Q}(h_1^{\vec{p}},...,h_m^{\vec{p}},g^{\vec{p}})}(y_1^{\vec{p}}, ..., y_n^{\vec{p}}) \geq n|\vec{p}| - n.$$

Together with equation (5.6.4), we have

$$\begin{aligned} tr.deg_{\mathbb{Q}(h_1^{\vec{p}},...,h_m^{\vec{p}})}(g^{\vec{p}}) &= tr.deg_{\mathbb{Q}(h_1^{\vec{p}},...,h_m^{\vec{p}})}(g^{\vec{p}}, y_1^{\vec{p}}, ..., y_n^{\vec{p}}) - tr.deg_{\mathbb{Q}(h_1^{\vec{p}},...,h_m^{\vec{p}},g^{\vec{p}})}(y_1^{\vec{p}}, ..., y_n^{\vec{p}}) \\ &\leq (n+1)|\vec{p}| - (n+1) - (n|\vec{p}| - n) \\ &= |\vec{p}| - 1. \end{aligned}$$

Hence by Proposition 5.6.3 there is $\gamma \in \Gamma$ such that $g^{\vec{p}}, h_1^{\vec{p}}, ..., h_m^{\vec{p}}, \gamma$ are multiplicatively dependent. Because H is closed under all power functions and contains all interpretation of constant symbols $\dot{\gamma}$, we get that $g \in H$.

□

Corollary 5.6.5 *Let $(M, G) \models \tilde{T}$ and H be a subgroup of G, which is closed under all definable power functions and contains all interpretations of the constants $\dot{\gamma}$, where $\gamma \in \Gamma$. If A is $\mathbf{cl}_T$-independent over G, then*

$$\mathbf{cl}_T(A, H) \cap G = H.$$

Proof: We use the same proof as in the case of Corollary 4.5.2. First note that H is obviously a subset of $\mathbf{cl}_T(A, H) \cap G$. By the Main Lemma 5.6.4 it is only left to show that

$$\mathbf{cl}_T(A, H) \cap G \subseteq \mathbf{cl}_T(H) \cap G. \tag{5.6.6}$$

So let $z \in \mathbf{cl}_T(A, H) \cap G$ and A' a minimal subset of A such that $z \in \mathbf{cl}_T(A', H) \cap G$. For a contradiction, suppose that A' is non-empty and let $a \in A$. By minimality of A', we have $z \notin \mathbf{cl}_T(A' - \{a\}, H)$. But then the Steinitz Exchange Principle implies that $a \in \mathbf{cl}_T(A' - \{a\}, z, H)$. Since $z \in G$, we get that

$$a \in \mathbf{cl}_T(A' - \{a\}, G).$$

But this is a contradiction to the $\mathbf{cl}_T$-independence of A over G. Hence A' is empty and $z \in \mathbf{cl}_T(H) \cap G$. Thus (5.6.6) holds.

□

Corollary 5.6.6 *Let $(M,G) \models \tilde{T}$ and H be a subgroup of G, which is closed under all definable power functions and contains all interpretations of the constants $\dot{\gamma}$, where $\gamma \in \Gamma$. If A is $\mathbf{cl}_T$-independent over G and $g \in G - (\mathbf{cl}_T(A,H))$, then*

$$\mathbf{cl}_T(A,H,g) \cap G = H\langle g \rangle := \{hg^k \mid h \in H, k \in \mathbb{Q}(\tau)\}.$$

Proof: Since $\mathbf{cl}_T(A,H,g)$ is a model of T and by $(KdmG5)$ the subgroup G is closed under all definable power functions, the inclusion $H\langle g \rangle \subseteq \mathbf{cl}_T(A,H,g) \cap G$ holds. Because $H\langle g \rangle$ is closed under all power functions, Corollary 5.6.5 implies that

$$H\langle g \rangle = \mathbf{cl}_T(A, H_G\langle g \rangle) \cap G \supseteq \mathbf{cl}_T(A,H,g) \cap G.$$

□

5.7 Proof of completeness

In this section we will give the proof of the completeness of $\tilde{T}$, where $\tilde{T}$ is either $T \cup KmdG$ or $T \cup KdmG'$. Note that the argument in the following proof is the same used in [vdDG06] and Chapter 4 of this text. It differs from that proof only by the fact that the groups considered here are divisible. This even simplifies the proof.

Let $\mathcal{N} := (M, G(N)), \mathcal{N}' := (M', G(N'))$ be two $(|\Gamma|)^+$-saturated models of $\tilde{T}$. Then M, M' are models of T. Let $\mathcal{E}$ be the set of all $\mathfrak{L}_\Gamma^\tau$-elementary maps from M to M'. Let $\mathcal{S}$ be the set of all $\beta \in \mathcal{E}$ such that there exist

- a finite subset A of M, and a finite subset A' of M',
- a subgroup H of $G(N)$ of cardinality at most $|\Gamma|$ and a subgroup H' of $G(N')$ of cardinality at most $|\Gamma|$

such that

1. β is an $\mathfrak{L}_\Gamma^\tau(G)$-isomorphism between $(\mathbf{cl}_T(A,H),H)$ and $(\mathbf{cl}_T(A',H'),H')$,
2. A is $\mathbf{cl}_T$-independent over $G(N)$, and A' is $\mathbf{cl}_T$-independent over $G(N')$ with $\beta(A) = A'$,
3. $\Gamma \leq H$, $\Gamma \leq H'$,
4. H, H' are closed under all power function definable in $\tilde{\mathbb{R}}$.

Note that by Corollary 5.6.5, $(\mathbf{cl}_T(A,H),H)$ rsp. $(\mathbf{cl}_T(A',H'),H')$ from the above definition is a $\mathfrak{L}_\Gamma^\tau(G)$-substructure of $(M,G(N))$ rsp. $(M',G(N'))$.

Lemma 5.7.1 *$\mathcal{S}$ is a back-and-forth-system of $\mathfrak{L}_\Gamma^\tau(G)$-isomorphisms.*

Proof: In order to prove this statement, we will show that for every $\beta \in \mathcal{S}$ and every $a \in \mathcal{N}$, there is a $\gamma \in \mathcal{S}$ such that γ extends β and $a \in dom(\gamma)$. In fact, this is enough, because of the symmetry of the setting.
Let $\beta \in \mathcal{S}$ and $a \in \mathcal{N}$. We can assume that $a \notin dom(\beta)$. Further let A, A', H, H' witness that $\beta \in \mathcal{S}$.
Case 1: $a \in G(N)$.
Let C be the cut of a in $\mathbf{cl}_T(A,H)$. Further let C' be the cut in $\mathbf{cl}_T(A',H')$ corresponding to C under β. Since $\mathcal{N}'$ is saturated, there are $p,q \in M'$ such that all elements in the interval (p,q) realize C'. Since $G(N')$ is dense in M', there is $a' \in G(N')$ and $a' \in (p,q)$. Hence a' realizes the cut C'. Since a' realize the cut C' and T is o-minimal, there is an isomorphism $\gamma : \mathbf{cl}_T(A,H,a) \to \mathbf{cl}_T(A',H',a')$ extending β with $\gamma(a) = a'$. We set

$$K := \mathbf{cl}_T(A,H,a) \cap G(N) \text{ and } K' := \mathbf{cl}_T(A',H',a') \cap G(N').$$

By Corollary 5.6.6, we have that $K = H\langle a\rangle$ and $K' = H'\langle a'\rangle$, and that K and K' are subgroups of $G(N)$ rsp. $G(N')$ which are closed under all power functions definable in $\tilde{\mathbb{R}}$. Thus we get for all $h \in G(N)$ that $h \in K$ iff $\gamma(h) \in K'$. Hence γ is an isomorphism between $(\mathbf{cl}_T(A,H,a),K)$ and $(\mathbf{cl}_T(A',H',a'),K')$.
Case 2: $a \in \mathbf{cl}_T(A,G(N))$.
Let $g_1,...,g_n \in G(N)$ such that $a \in \mathbf{cl}_T(A,\{g_1,...,g_n\})$. By using Case 1 n times, we get a $\mu \in \mathcal{S}$ such that $g_1,...,g_n \in dom(\mu)$ and $A \subseteq dom(\mu)$. Since $dom(\mu)$ is a model of T, we have $a \in dom(\mu)$ with $\mu \in \mathcal{S}$.
Case 3: Otherwise, ie. $a \notin \mathbf{cl}_T(A,G(N))$.
As in Case 1, let C be the cut of a in $\mathbf{cl}_T(A,H)$ and let C' be the corresponding cut of C under β in $\mathbf{cl}_T(A',H')$. Again by saturation, we can assume that there are $p,q \in \mathcal{N}'$ such that every element in the interval (p,q) realizes the cut C'. Let $\overline{d}$ be the set A written as a tuple. Let $f_1,...,f_n$ 0-definable functions in T. By axiom $(\mathrm{KdmG7}_{n,(f_1,...,f_n)})$, we know that there is a $b \in (p,q)$ such that for $i = 1,...,n$ and every tuple $\overline{g}$ of elements of $G(N')$

$$f_i(\bar{g},\bar{d}) \neq b.$$

Thus by saturation, there is an $a' \in (p,q)$ such that $a' \notin \mathbf{cl}_{T'}(A',G(N'))$. Since a' realizes the cut C', there is an $\mathfrak{L}_\Gamma^\tau$-isomorphism γ from $\mathbf{cl}_T(A,a,H)$ to $\mathbf{cl}_T(A',a',H')$

extending β and sending a to a'. Since $a \notin \mathbf{cl}_T(A, G(N))$ and $a' \notin \mathbf{cl}_T(A', G(N'))$, we get that

$$\mathbf{cl}_T(A, a, H) \cap G(N) = H \text{ and } \mathbf{cl}_T(A', a', H') \cap G(N') = H'.$$

Since $\beta(H) = H'$ and γ extends β, we get that γ is an $\mathfrak{L}^\tau_\Gamma$-isomorphism from $(\mathbf{cl}_T(A, a, H), H)$ to $(\mathbf{cl}_T(A', a', H'), H')$. Since $\beta(A \cup \{a\}) = A' \cup \{a'\}$, we have $\beta \in \mathcal{S}$.

□

Theorem 5.7.2 *$\tilde{T}$ is complete.*

Proof: It is enough to show that $\mathcal{S}$ is non-empty. Let $P := \mathbf{cl}_T(\emptyset) \subseteq M$ and $P' := \mathbf{cl}_T(\emptyset) \subseteq M'$. Since $M, M' \models T$, there is an $\mathfrak{L}^\tau_\Gamma$-isomorphism γ between (P, Γ) and (P', Γ). Hence γ is an $\mathfrak{L}^\tau_\Gamma(G)$-isomorphism and $\gamma \in \mathcal{S}$.

□

5.8 Proof of near model completeness

Definition 5.8.1 *(i) Define* the language $\mathfrak{L}^\tau_\Gamma(G)^+$ *as the language* $\mathfrak{L}^\tau_\Gamma(G)$ *together with predicates* P_ϕ *for every* $\mathfrak{L}^\tau_\Gamma(G)$*-formula* ϕ *of the form*

$$\exists \overline{y} \exists \overline{z}\ \psi(\overline{x}, \overline{y}, \overline{z}) \wedge \bigwedge_i G(z_i), \tag{5.8.1}$$

where ψ *is a quantifier-free* $\mathfrak{L}^\tau_\Gamma$*-formula.*
(ii) the $\mathfrak{L}^\tau_\Gamma(G)^+$-theory $(T \cup \mathrm{KdmG})^+$ *is the theory* $T \cup KdmG$ *together with the following axioms: for every* $\mathfrak{L}^\tau_\Gamma(G)$*-formula* ϕ *of the form* (5.8.1) *add the axiom*

$$P_\phi(\overline{x}) \leftrightarrow \phi(\overline{x}).$$

For showing near model completeness of $T \cup \mathrm{KdmG}$, it is therefore to show

Theorem 5.8.2 *$(T \cup KdmG)^+$ has quantifier elimination.*

Let $\mathfrak{L}_g$ be the language of ordered groups. Let $\mathfrak{L}_{g,\Gamma}$ be $\mathfrak{L}_g$ augmented by a constant symbol $\dot{\gamma}$ for every $\gamma \in \Gamma$. Remember from Chapter 4 that for a $\mathfrak{L}_{g,\Gamma}$-formula φ, we say that the G-restriction of φ, φ_G is the $\mathfrak{L}^\tau_\Gamma(G)$-formula inductively defined by

$$\begin{aligned}
\varphi_G &:= \varphi && \text{if } \varphi \text{ is atomic,}\\
\varphi_G &:= \neg\psi_G && \text{if } \varphi = \neg\psi,\\
\varphi_G &:= \chi_G \wedge \psi_G && \text{if } \varphi = \chi \wedge \psi,\\
\varphi_G &:= \exists x (G(x) \wedge \psi_G) && \text{if } \varphi = \exists x \psi.
\end{aligned}$$

Proposition 5.8.3 *Let $\varphi(x)$ be an $\mathfrak{L}_{g,\Gamma}$-formula, then φ_G is equivalent to a quantifier-free $\mathfrak{L}_\Gamma^\tau(G)$-formula.*

Proof: This directly follows from quantifier elimination for ordered divisible abelian groups G. (See e.g. [M02] Theorem 3.1.17)

$\square$

Proof of Theorem 5.8.2: In order to show that $(T \cup \text{KdmG})^+$ has quantifier-elimination, let $\vec{a} = (a_1, ..., a_m) \in \mathcal{N}$ and $\vec{b} = (b_1, ..., b_m) \in \mathcal{N}'$ having the same quantifier-free $(\mathfrak{L}_\Gamma^\tau(G))^+$-type. We will now show that then there is an element in the back-and-forth-system $\mathcal{S}$ sending $\vec{a}$ to $\vec{b}$. This implies that $\vec{a}$ and $\vec{b}$ have the same $(\mathfrak{L}_\Gamma^\tau(G))^+$-type and hence that $(T \cup KdmG)^+$ has quantifier-elimination.
Let $(a_1, ..., a_r)$ be $\mathbf{cl}_T$-independent over $\mathbf{cl}_T(G(N))$.

(1) $(b_1, ..., b_r)$ is $\mathbf{cl}_T$-independent over $\mathbf{cl}_T(G(N'))$.

Proof: For a contradiction, suppose $(b_1, ..., b_r)$ is $\mathbf{cl}_T$-dependent over $\mathbf{cl}_T(G(N'))$. Then there are formulas $\mathfrak{L}_\Gamma^\tau$-formula φ, ψ without parameters and there are $g_1, ..., g_n \in G(N')$ such that
$\psi(\bar{x}, \bar{y})$ is the formula

$$\varphi(\bar{x}, \bar{y}) \wedge \forall z \; ((\varphi(z, x_2, ..., x_r, \bar{y})) \rightarrow (z = x_1))$$

and $\mathcal{N}' \models \psi(b_1, ..., b_r, g_1, ..., g_n)$. By model-completeness of T, we can assume that ψ is of the form

$$\exists \bar{u} \chi(\bar{x}, \bar{y}, \bar{u}),$$

where χ is a quantifier-free $\mathfrak{L}_\Gamma^\tau$-formula. Hence

$$\mathcal{N}' \models \exists \bar{y} \exists \bar{u} \chi(b_1, ..., b_r, \bar{y}, \bar{u}) \wedge \bigwedge_{i=1}^{n} G(y_i).$$

Since $\vec{a}$ and $\vec{b}$ have the same quantifier-free $\mathfrak{L}_\Gamma^\tau(G)^+$-type,

$$\mathcal{N} \models \exists \bar{y} \exists \bar{u} \chi(a_1, ..., a_r, \bar{y}, \bar{u}) \wedge \bigwedge_{i=1}^{n} G(y_i).$$

But this contradicts the $\mathbf{cl}_T$-independence of $(a_1, ..., a_r)$ over $\mathbf{cl}_T(G(N))$.

$\square(1)$

By symmetry, we can take r maximal such that, perhaps after rearranging the a_i's, $(a_1, ..., a_r)$ is a maximal such independent subtupel of $\vec{a}$ and $(b_1, ..., b_r)$ is a maximal such independent subtupel of $\vec{b}$. Now let $g_1, ..., g_l \in G(N)$ such that

$$a_{r+1}, ..., a_m \in \mathbf{cl}_T(\{a_1, ..., a_r, g_1, ..., g_l\}).$$

Suppose $\varphi_1(\bar{y}), ..., \varphi_n(\bar{y})$ are $\mathfrak{L}_{g,\Gamma}$-formulas and $\psi_1(\bar{x}, \bar{y}), ..., \psi_t(\bar{x}, \bar{y})$ are $\mathfrak{L}_\Gamma^\tau$-formulas such that

$$\mathcal{N} \models \bigwedge_{i=1}^{n} \varphi_{iG}(g_1, ..., g_l) \wedge \bigwedge_{i=1}^{t} \psi_i(a_1, ..., a_m, g_1, ..., g_l) \wedge \bigwedge_{i=1}^{l} G(g_i).$$

Since T is model complete and because of Proposition 5.8.3, we can assume that the formula

$$\bigwedge_{i=1}^{n} \varphi_{iG}(\bar{y}) \wedge \bigwedge_{i=1}^{t} \psi_i(\bar{x}, \bar{y}) \wedge \bigwedge_{i=1}^{l} G(y_i)$$

is of the form

$$\exists \bar{w} \chi(\bar{w}, \bar{x}, \bar{y}) \wedge \bigwedge_{i=1}^{l} G(y_i),$$

where χ is quantifier-free $\mathfrak{L}_\Gamma^\tau$-formula. Thus

$$\mathcal{N} \models \exists \bar{y} \exists \bar{w} \chi(\bar{w}, a_1, ..., a_m, y_1, ..., y_l) \wedge \bigwedge_{i=1}^{l} G(y_i).$$

Since $(a_1, ..., a_m)$ and $(b_1, ..., b_m)$ have the same quantifier-free $(\mathfrak{L}_\Gamma^\tau(G))^+$-type, we have

$$\mathcal{N}' \models \exists \bar{y} \exists \bar{w} \chi(\bar{w}, b_1, ..., b_m, y_1, ..., y_l) \wedge \bigwedge_{i=1}^{l} G(y_i).$$

Hence by saturation we get that there are $h_1, ..., h_r \in G(N')$ such that for every $\mathfrak{L}_{g,\Gamma}$-formula $\varphi(\bar{y})$ and every $\mathfrak{L}_\Gamma^\tau$-formula $\psi(\bar{y}, \bar{z})$,

$$\mathcal{N} \models \varphi_G(g_1, ..., g_l) \text{ iff } \mathcal{N}' \models \varphi_G(h_1, ..., h_l)$$
$$\mathcal{N} \models \psi(g_1, ..., g_l, a_1, ..., a_m) \text{ iff } \mathcal{N}' \models \psi(h_1, ..., h_l, b_1, ..., b_m).$$

Hence there is an $\mathfrak{L}_\Gamma^\tau$-elementary map β with

$$\begin{aligned} \beta : \mathbf{cl}_T(\{a_1, ...a_r, g_1, ..., g_l\}) &\to \mathbf{cl}_T(\{b_1, ...b_r, h_1, ..., h_l\}) \\ a_i &\mapsto b_i, \quad \text{for } i = 1, ..., m \\ g_i &\mapsto h_i, \quad \text{for } i = 1, ..., l \end{aligned}$$

and $\beta(\Gamma\langle g_1, ..., g_l\rangle) = \Gamma\langle h_1, ..., h_l\rangle$. Since $\{a_1, ..., a_r\}$ is $\mathbf{cl}_T$-independent over $G(N)$ and $\{b_1, ..., b_r\}$ is $\mathbf{cl}_T$-independent over $G(N')$, we get that $\beta \in \mathcal{S}$.

□

Chapter 6

Induced structure and o-minimal open core

This chapter considers a few corollaries from the work done in Chapter 4 and Chapter 5. We will start by analyzing the induced structure on Γ in the structure $(\tilde{\mathbb{R}}, \Gamma)$. In fact, it will be shown that for divisible Γ this structure will be weakly o-minimal. Further we will show that all the structures $(\tilde{\mathbb{R}}, \Gamma)$ we have examined so far, have o-minimal open core. We will see that the result directly follows from the results of Chapter 4 and Chapter 5 and from the work Berenstein, Ealy and Günaydin done in [BEG07].

6.1 Basic definitions and notation

In this section we will give the definition of weak o-minimality and open core.

Definition 6.1.1 *Let $\mathcal{R} = (R, <, ...)$ be an expansion of a dense linear order $(R, <)$ without endpoints. $\mathcal{R}$ is* weakly o-minimal*, if every definable subset of R is a finite union of convex sets.*

Of course, every o-minimal structure is weakly o-minimal. Now consider the real field $\overline{\mathbb{R}}$ together with a dense multiplicative subgroup Γ which is divisible and has the Mann property. In [vdDG06] van den Dries and Günaydin proved that the group Γ with the structure induced by $(\overline{\mathbb{R}}, \Gamma)$ is weakly o-minimal. In Section 2, it will be shown by the same method that this also holds in the cases considered in Chapter 4 and 5.

Definition 6.1.2 *Let $\mathcal{R} = (R, <, ...)$ be an expansion of a dense linear order $(R, <)$ without endpoints. We call the structure $(R, (U))$, where U ranges over all open, definable sets in $\mathcal{R}$ of all arities, the* open core of $\mathcal{R}$. *We say $\mathcal{R}$* has o-minimal open core*, if the open core of $\mathcal{R}$ is o-minimal.*

For results on o-minimal open cores, see for example [MS99] and [DMS08]. Further note that Berenstein, Ealy and Günaydin showed in [BEG07] that the real field together with a dense multiplicative subgroup having the Mann property has o-minimal open core. Again, we will use their work in Section 3 of this chapter to show that this also holds in the cases we have considered so far.

Let $\tilde{T}$ be one of $T \cup \text{dmG}$, $T \cup \text{KmdG}$ or $T \cup \text{KdmG'}$.

Definition 6.1.3 *Let $(M, G) \models \tilde{T}$. We say that a subgroup H of G is* $\mathbb{Q}(\tau)$-pure in *G, if for every $h \in H$ and $p \in \mathbb{Q}(\tau)$*

$$h^p \in H \textit{ iff } h^p \in G.$$

Further note that in the case $\tilde{T}$ being $T \cup \mathrm{dmG}$, H is $\mathbb{Q}(\tau)$-pure in G iff H is pure in G. For $\tilde{T}$ being either $T \cup \mathrm{KmdG}$ or $T \cup \mathrm{KdmG'}$, H is $\mathbb{Q}(\tau)$-pure in G iff it is closed under all definable power functions.

Definition 6.1.4 *Let $(M, G) \models \tilde{T}$, H be a $\mathbb{Q}(\tau)$-pure subgroup of G and $\vec{g} := (g_1, ..., g_n) \in G^n$. We define the $\mathbb{Q}(\tau)$-closure of H and $\vec{g}$ in G by*

$$H_G^\tau\langle\vec{g}\rangle := \{(h \cdot \prod_{i=1}^{n} g_i^{p_i})^q \mid h \in H, \vec{p} \in \mathbb{Q}(\tau)^n, q \in \mathbb{Q}(\tau), h \cdot \prod_{i=1}^{n} g_i^{p_i} \in G^{[q]}\}.$$

Obviously, this definition implies the following lemma.

Lemma 6.1.5 *Let $(M, G) \models \tilde{T}$, H be a $\mathbb{Q}(\tau)$-pure subgroup of G and $\vec{g} \in G^n$. Then $H_G^\tau\langle\vec{g}\rangle$ is $\mathbb{Q}(\tau)$-pure in G.*

6.2 Induced structure

Let $m \in \mathbb{Q}(\tau)$ and $\vec{k} = (k_1, .., k_n) \in \mathbb{Q}(\tau)^n$. Let $D_{m,\vec{k}}(x_1, ..., x_n)$ be the $\mathfrak{L}_\Gamma^\tau(G)$-formula

$$\bigwedge_{i=1}^{n} G(x_i) \wedge \exists y\, G(y) \wedge x_1^{k_1} \cdot ... \cdot x_n^{k_n} = y^m.$$

We also write $D_{m,\vec{k}}$ for the set defined by the above formula. Note that $D_{m,\vec{k}}$ is the set

$$\{(g_1, ..., g_n) \in G^n \mid g_1^{k_1} \cdot ... \cdot g_n^{k_n} \in G^{[m]}\}.$$

Obviously, $(G^{[m]})^n$ is a subset of $D_{m,\vec{k}}$. Since $(G^{[m]})^n$ is dense, $D_{m,\vec{k}}$ is dense as well.

Lemma 6.2.1 *Let $(\mathcal{R}, \mathcal{G}) \models T \cup dmG_\Gamma$. Let $q \in \mathbb{Q}(\tau)$ and $\vec{p} \in \mathbb{Q}(\tau)^n$. Then there are $s \in \mathbb{N}$, $\vec{a_0} \in \mathbb{Z}^n$ and a $\mathfrak{L}^\tau$-formula $\varphi(x_1, ..., x_n)$ such that*

$$D_{q,\vec{p}} = D_{s,\vec{a_0}} \cap \{\vec{x} \in \mathcal{R} \mid (\mathcal{R}, \mathcal{G}) \models \varphi(\vec{x})\}.$$

<u>Proof:</u> Let $q \in \mathbb{Q}(\tau)$, $\vec{p} \in \mathbb{Q}(\tau)^n$ and $\vec{g} \in \mathcal{G}$ with $\vec{g} \in D_{q,\vec{p}}$. Since

$$g_1^{p_1} \cdot ... \cdot g_n^{p_n} = h^q \text{ iff } g_1^{\frac{p_1}{q}} \cdot ... \cdot g_n^{\frac{p_n}{q}} = h,$$

we can assume that $q \in \mathbb{N}$. Further suppose $1, t_1, ..., t_l$ is a basis of the $\mathbb{Q}$-vector space generated by $1, p_1, ..., p_n$. Let $a_{i,j} \in \mathbb{Q}$ be such that

$$p_i = a_{i,0} + a_{i,1}t_1 + ... + a_{i_l}t_l.$$

Then

$$g_1^{p_1} \cdot ... \cdot g_n^{p_n} = h^q \text{ iff } g_1^{a_{1,0}} g_1^{a_{1,1}t_1} \cdot ... \cdot g_1^{a_{1,l}t_l} \cdot g_2^{a_{2,0}} \cdot ... \cdot g_n^{a_{n,l}t_l} = h^q.$$

By multiplying all exponents on each side by the product of denominators of the $a_{i,j}$'s, we can assume that $a_{i,j} \in \mathbb{Z}$ for all $i = 1, ..., n$ and $j = 0, ..., l$. Then $g_i^{a_{i,j}} \in G$ for all $i = 1, .., n$ and $j = 0, ...l$, and further

$$g_1^{a_{1,1}t_1} \cdot ... \cdot g_1^{a_{1,l}t_l} \cdot g_2^{a_{2,1}t_1} \cdot ... \cdot g_n^{a_{n,l}t_l} \in G^{[t_1]} \cdot ... \cdot G^{[t_n]} \cap G.$$

Suppose

$$g_1^{a_{1,1}t_1} \cdot ... \cdot g_1^{a_{1,l}t_l} \cdot g_2^{a_{2,1}t_1} \cdot ... \cdot g_n^{a_{n,l}t_l} \neq 1$$

Hence $1, t_1, ..., t_l$ must be $\mathbb{Q}$-linearly dependent by (dmG7). But this a contradiction. Hence we have that

$$g_1^{a_{1,1}t_1} \cdot ... \cdot g_1^{a_{1,l}t_l} \cdot g_2^{a_{2,1}t_1} \cdot ... \cdot g_n^{a_{n,l}t_l} = 1 \tag{6.2.1}$$

and

$$g_1^{a_{1,0}} \cdot ... \cdot g_n^{a_{n,0}} = h^q.$$

Let $\varphi(x_1, ..., x_n)$ be the $\mathfrak{L}^\tau$-formula given by (6.2.1). Let $\vec{a_0} := (a_{1,0}, ..., a_{n,0})$. So the above implies that

$$\prod_{i=1}^n g_i^{p_i} \in G^{[q]} \text{ iff } \prod_{i=1}^n g_i^{a_{i,0}} \in G^{[q]} \text{ and } \varphi(g_1, ..., g_n).$$

Hence we get that

$$D_{q,\vec{p}} = D_{q,\vec{a_0}} \cap \{\vec{x} \in \mathcal{R} \mid (\mathcal{R}, \mathcal{G}) \models \varphi(\vec{x})\}.$$

□

In Chapter 4 and 5 we have proven the Main Lemma for $\tilde{T}$. This generalizes to the following statement:

Theorem 6.2.2 *Let $(\mathcal{R}, \mathcal{G}) \models \tilde{T}$ and H be a $\mathbb{Q}(\tau)$-pure subgroup of $\mathcal{G}$ containing all interpretations of the constants $\dot{\gamma}$, where $\gamma \in \Gamma$. If A is $\boldsymbol{cl}_T$-independent over $\mathcal{G}$ and $\vec{g} \in (\mathcal{G} - \boldsymbol{cl}_T(A, H))^n$, then*

$$\boldsymbol{cl}_T(A, H, \vec{g}) \cap \mathcal{G} = H^\tau_{\mathcal{G}}\langle \vec{g} \rangle.$$

Proof: For $\tilde{T} = T \cup \text{KdmG}$ the statement follows from Corollary 5.6.6 and the fact that $\mathcal{G}$ is closed under all power functions. In the case of $\tilde{T} = T \cup \text{dmG}_\Gamma$ note that by Lemma 6.2.1

$$H^\tau_{\mathcal{G}}\langle\vec{g}\rangle \subseteq \{(h \cdot \vec{g}^{\vec{p}})^q \mid h \in H, \vec{p} \in \mathbb{Z}^n, q \in \mathbb{N}, h \cdot \vec{g}^{\vec{p}} \in G^{[q]}\}.$$

Hence Corollary 4.5.3 implies the statement in this case.

□

The following Theorem is a generalization of Proposition 53 from [BEG07] and Theorem 7.5 from [vdDG06]. In their proof Berenstein, Ealy and Günaydin use Lemma 5.12 of [vdDG06]. We will use Main Lemma 6.2.2 instead. Let $(\mathcal{R}, \mathcal{G})$ be a model of $\tilde{T}$.

Theorem 6.2.3 *Let $X \subset \mathcal{G}^n$ be definable in $(\mathcal{R}, \mathcal{G})$. Then X is a boolean combination of sets of the form $Y \cap \vec{g}D_{m,\vec{k}}$, where Y is an $\mathfrak{L}^\tau_\Gamma$-definable set, $\vec{g} \in \Gamma^n$, $m \in \mathbb{Q}(\tau)$ and $\vec{k} \in \mathbb{Q}(\tau)^n$.*

Proof: Let $\mathcal{N} = (M, G(N)), \mathcal{N}' = (M', G(N'))$ be two $|\mathcal{R}|^+$-saturated elementary extensions of $(\mathcal{R}, \mathcal{G})$. Let $\mathcal{S}$ be the back and forth system of $\mathfrak{L}^\tau_\Gamma(G)$-isomorphisms between $\mathcal{N}$ and $\mathcal{N}'$ constructed in Chapter 4, Section 5 and Chapter 5, Section 6. Let $\vec{g} \in G(N)^n$ and $\vec{h} \in G(N')^n$ be such that for every $\mathfrak{L}^\tau_\Gamma$-formula φ with parameters in $\mathcal{R}$, and all $m, \vec{k} \in \mathbb{Q}(\tau)$ and $\gamma \in \mathcal{G}$

$$\mathcal{N} \models \varphi(\vec{g}) \wedge D_{m,(1,\vec{k})}(\gamma, \vec{g}) \text{ iff } \mathcal{N}' \models \varphi(\vec{h}) \wedge D_{m,(1,\vec{k})}(\gamma, \vec{h}). \qquad (6.2.2)$$

Now by [H93] Theorem 8.4.1, it is only left to show there is $\beta \in \mathcal{S}$ such that β maps $\vec{g}$ to $\vec{h}$.

Since $\vec{g}$ and $\vec{h}$ satisfy the same $\mathfrak{L}^\tau_\Gamma$-type over $\mathcal{R}$, there is an $\mathfrak{L}^\tau_\Gamma$-isomorphism β over $\mathcal{R}$ mapping $\vec{g}$ to $\vec{h}$. So we only need to show that β is in $\mathcal{S}$. First let C be a subset of $\mathcal{R}$ such that

$$\mathbf{cl}_T(C, \mathcal{G}) = \mathcal{R} \text{ and } C \text{ is } \mathbf{cl}_T\text{-independent over } \mathcal{G}.$$

Since $\mathcal{N}, \mathcal{N}'$ are elementary extensions of $(\mathcal{R}, \mathcal{G})$, C is also $\mathbf{cl}_T$-independent over $G(N)$ and $G(N')$. Hence note by Theorem 6.2.2 that

$$\mathbf{cl}_T(C, \vec{g}) \cap G(N) = \mathcal{G}^\tau_{G(N)}\langle\vec{g}\rangle,$$

and that this group is $\mathbb{Q}(\tau)$-pure. Further note that $\beta(\mathcal{G}_{G(N)}\langle\vec{g}\rangle) = \mathcal{G}_{G(N')}\langle\vec{h}\rangle$, since by (6.2.2) for all $\gamma \in \mathcal{G}$, $\vec{k} \in \mathbb{Q}(\tau)^n$ and $m \in \mathbb{Q}(\tau)$

$$(\gamma \prod_{i=1}^{n} g_i^{k_i}) \in G(N)^{[m]} \text{ iff } (\gamma \prod_{i=1}^{n} h_i^{k_i}) \in G(N')^{[m]}.$$

Hence $\beta \in \mathcal{S}$.

□

We directly get the following corollaries.

Corollary 6.2.4 *Let Γ satisfy (G1)-(G4) and let $X \subset \mathcal{G}^n$ be definable in $(\mathcal{R}, \mathcal{G})$. Then X is a boolean combination of sets of the form $Y \cap \vec{g}D_{m,\vec{k}}$, where Y is an $\mathfrak{L}_\Gamma^\tau$-definable set, $\vec{g} \in \Gamma^n$, $m \in \mathbb{N}$, and $\vec{k} \in \mathbb{Z}^n$.*

Proof: First note that the only difference to Theorem 6.2.3 is that m is in $\mathbb{N}$ and not just in $\mathbb{Q}(\tau)$ and that $\vec{k}$ is a tuple with coordinates in $\mathbb{Z}$ and not just in $\mathbb{Q}(\tau)$. This implies that the statement directly follows from Lemma 6.2.1 and Theorem 6.2.3.

□

Corollary 6.2.5 *Let Γ be divisible and satisfy (G1)-(G4). Then $X \subseteq \mathcal{G}^n$ is definable in $(\mathcal{R}, \mathcal{G})$ if and only if $X = Y \cap G^n$ for some $\mathfrak{L}_\Gamma^\tau$-definable Y. Further $\mathcal{G}$ with the induced structure by $(\mathcal{R}, \mathcal{G})$ is weakly o-minimal.*

Proof: Because Γ is divisible, for every $m \in \mathbb{N}$ and $\vec{k} \in \mathbb{Z}^{|\vec{k}|}$, $D_{m,\vec{k}}$ is $\mathcal{G}^{|\vec{k}|}$. Hence the statement directly follows from the previous corollary.

□

Corollary 6.2.6 *Let Γ satisfy (H1)-(H2). Then $X \subseteq \mathcal{G}^n$ is definable in $(\mathcal{R}, \mathcal{G})$ if and only if $X = Y \cap \mathcal{G}^n$ for some $\mathfrak{L}_\Gamma^\tau$-definable Y. Further $\mathcal{G}$ with the induced structure by $(\mathcal{R}, \mathcal{G})$ is weakly o-minimal.*

Proof: Remember that (H1) says that Γ is closed under all power functions. Hence for every $m \in \mathbb{Q}(\tau)$ and $\vec{k} \in \mathbb{Q}(\tau)^{|\vec{k}|}$, $D_{m,\vec{k}}$ is just $\mathcal{G}^{|\vec{k}|}$.

□

Further, we are now able to solve a question raised in Chapter 5. In the introduction to Chapter 5, the question was asked whether the subgroup $2^{\mathbb{Q}}$ is definable in the structure $(\tilde{\mathbb{R}}, 2^{\mathbb{Q}(\tau)})$. Using the above results, we can show that it is not, at least under Conjecture 5.3.3 and Condition 3.1.2 for τ.

Theorem 6.2.7 *Assume Conjecture on intersection with tori 5.3.3 and Condition 3.1.2 for τ. Then $2^{\mathbb{Q}}$ is not definable in $(\tilde{\mathbb{R}}, 2^{\mathbb{Q}(\tau)})$.*

Proof: By Corollary 6.2.7, every definable subset of $2^{\mathbb{Q}(\tau)}$ is a finite union of convex subsets. Suppose for a contradiction that $2^{\mathbb{Q}}$ is definable in $(\tilde{\mathbb{R}}, 2^{\mathbb{Q}(\tau)})$. Then there are convex subsets $C_1, ..., C_n$ such that $\bigcup_{i=1}^{n} C_i \cap 2^{\mathbb{Q}(\tau)} = 2^{\mathbb{Q}}$. Then there is $j \in \{1, ..., n\}$ such that there are $x_1, x_2 \in C_j$ with $x_1 < x_2$. Since C_j is convex, we have that the interval (x_1, x_2) of $2^{\mathbb{Q}(\tau)}$ is a subset of C_j. But then there is $y \in 2^{\mathbb{Q}}$ such that $y^{\tau} \in (x_1, x_2)$, since $(2^{\mathbb{Q}})^{[\tau]}$ is dense in $\mathbb{R}^{>0}$. This is a contradiction to $C_j \cap 2^{\mathbb{Q}(\tau)} \subseteq 2^{\mathbb{Q}}$.

□

6.3 O-minimal open core

In this section, we will show that the structure $(\tilde{\mathbb{R}}, \Gamma)$ has o-minimal open core. Again if Γ is a dense multiplicative subgroup satisfying (H1) and (H2), we assume that either

(i) τ is algebraic and the Uniform Schanuel Condition 3.1.4 holds for τ, or

(ii) τ is arbitrary, Schanuel Condition 3.1.2 holds for τ, the Conjecture on intersection with tori 5.3.3 holds and, further, the set I witnessing (H1) for Γ is finite.

In order to show that in these cases the considered structures have o-minimal open core, we will use a result from [BEG07]. The following Theorem is an instance of their Corollary 49.

Theorem 6.3.1 *Let Γ be any dense multiplicative subgroup of the real field. If*

(*) $(\tilde{\mathbb{R}}, \Gamma)$ is near model complete and

(**) for each $(\mathcal{R}, \mathcal{G}) \models Th(\tilde{\mathbb{R}}, \Gamma)$, each $\vec{a} \in \mathcal{R}^{|\vec{a}|}$ and $\mathbb{D} \subseteq \mathcal{G}^n$, with $\mathbb{D}$ being $\mathfrak{L}_{\Gamma}^{\tau}(G)$-definable over $\vec{a}$, there are an $\mathfrak{L}_{\Gamma}^{\tau}$-definable set $\mathbb{E}$ and an $\mathfrak{L}_{\Gamma}^{\tau}(G)$-definable set $\mathbb{S}$, which is dense in $\mathcal{G}^n$, such that $\mathbb{E} \cap \mathbb{S} = \mathbb{D}$, and furthermore, if $n = 1$, $\mathbb{D}$ is a finite union of such $\mathbb{E} \cap \mathbb{S}$, where $\mathbb{S}$ is 0-definable,

then $(\tilde{\mathbb{R}}, \Gamma)$ has o-minimal open core.

First note that Theorem 5.8.2, Corollary 6.2.6 and Theorem 6.3.1 imply the following.

Corollary 6.3.2 *Let Γ satisfies (H1) and (H2) and τ satisfies (i) or (ii). Then $(\tilde{\mathbb{R}}, \Gamma)$ has o-minimal open core.*

So for the following, suppose that Γ satisfies (G1)-(G4). The argument now presented is in essence the argument given in [BEG07] Section 5.

Lemma 6.3.3 *Let* $(\mathcal{R},\mathcal{G}) \models T \cup dmG_\Gamma$ *and* $\mathbb{D} \subset \mathcal{G}^n$ *be definable in* $(\mathcal{R},\mathcal{G})$. *Then there is* $d \in \mathbb{N}$ *such that* $\mathbb{D}$ *is a finite* union *of sets of the form* $\mathbb{E} \cap \vec{g}(G^{[d]})^n$, *where* $\vec{g} \in \Gamma^n$ *and* $\mathbb{E}$ *is definable in* $\mathcal{R}$.

Proof: By Corollary 6.2.4, we can assume that $\mathbb{D}$ is a finite boolean combination of sets of the form $\mathbb{E} \cap \vec{g}D_{m,\vec{k}}$, where $\mathbb{E}$ is definable in $\mathcal{R}$, $m \in \mathbb{N}$ and $\vec{k} \in \mathbb{Z}^n$. Because $(G^{[m]})^n$ is a subgroup of $D_{m,\vec{k}}$, $D_{m,\vec{k}}$ has finite index in G^n, since $(G^{[m]})^n$ has. Hence $\vec{g}D_{m,\vec{k}}$ can be replaced by $\vec{\gamma}(G^{[m]})^n$ for some $\vec{\gamma} \in \Gamma^n$. Also note that as in Proposition 4.6.3 the $\mathcal{R} - \vec{\gamma}(G^{[m]})^n$ is just the set

$$\bigcup_{\vec{\gamma_i} \neq \vec{\gamma}} \vec{\gamma_i}(G^{[m]})^n, \text{ where } \vec{\gamma_1} \cdot G^{[m]}, ..., \vec{\gamma_l} \cdot G^{[m]} \text{ are the all the cosets of } (G^{[m]})^n \text{ in } G^n.$$

Further note that if t is the least common multiple of m_1 and m_2, then $G^{[m_1]} \cap G^{[m_2]} = G^{[t]}$. This all implies that $\mathbb{D}$ can be written as the union of sets of the form

$$\mathbb{E} \cap \vec{\gamma}(G^{[d]})^n, \text{ where } d \in \mathbb{N}, \vec{\gamma} \in \Gamma^n \text{ and } \mathbb{E} \text{ is definable in } \mathcal{R}. \tag{6.3.1}$$

It is only left to show that we can choice d to be the same for all the sets in the union. Therefore just note that for m dividing t, $G^{[m]} = \bigcup_{\gamma_i \in G^m} \gamma_i G^{[t]}$, where $\gamma_1 \cdot G^{[t]}, ..., \gamma_l \cdot G^{[t]}$ are all the cosets of $G^{[t]}$ and $\gamma_i \in \Gamma$.

□

Note that for $d \in \mathbb{N}$ and $\vec{g}, \vec{h} \in \mathcal{G}$, the two sets $\vec{g}(G^{[d]})^m$ and $\vec{h}(G^{[d]})^m$ are either the same or their intersection is empty. Since $(G^{[d]})^m$ is dense in $\mathcal{G}$, both these sets are dense in $\mathcal{G}$ as well.

Lemma 6.3.4 *Let* $(\mathcal{R},\mathcal{G}) \models T \cup dmG_\Gamma$ *and let* $\mathbb{S}_i \subset \mathcal{G}^n$ *be sets which are dense in* $\mathcal{G}^n$, *and let* $A_i \subset \mathcal{R}^n$ *be* $\mathfrak{L}_\Gamma^\tau$*-definable sets with* $\bigcap A_i = \emptyset$. *Then* $\bigcup_{i=1}^m \mathbb{S}_i - A_i$ *is dense* $\mathcal{G}$.

Proof: Set $\mathbb{S} := \bigcup_{i=1}^m \mathbb{S}_i - A_i$. Let $\mathbb{U}$ be an open ball in $\mathcal{R}_{>0}^n$. By the cell composition Theorem 2.1.6 for o-minimal structures, and since $\bigcap A_i = \emptyset$, we can assume that there is an $i \in \{1, ..., m\}$ such that $\mathbb{U} \cap A_i = \emptyset$. Suppose for a contradiction that $\mathbb{S} \cap \mathbb{U}$ is empty. Then we get for every $i \in \{1, ..., m\}$ that $\mathbb{S}_i \cap \mathbb{U} \subset A_i$. But by the above, this implies that there is an $i \in \{1, ..., m\}$ with $\mathbb{S}_i \cap \mathbb{U} = \emptyset$. This is a contradiction to $\mathbb{S}_i$ being dense in $\mathcal{G}^n$.

□

Theorem 6.3.5 *Assume Condition 3.1.4 holds for* τ. *Let* Γ *satisfies (G1)-(G4). Then* $(\tilde{\mathbb{R}}, \Gamma)$ *has o-minimal open core.*

Proof: We only need to show that the Conditions $(*)$ and $(**)$ of Theorem 6.3.1 are satisfied. Condition $(*)$ holds by Theorem 4.6.4. Therefore it is only left to show that Condition $(**)$ holds as well. Let $(\mathcal{R}, \mathcal{G}) \models T \cup \mathrm{dmG}_\Gamma$ and $\mathbb{D} \subset \mathcal{G}^n$ be definable in $(\mathcal{R}, \mathcal{G})$. We first need to show that $\mathbb{D}$ is of the form $\mathbb{E} \cap \mathbb{S}$, where $\mathbb{E}$ is definable in $\mathcal{R}$ and $\mathbb{S}$ is dense in $\mathcal{G}^n$. By Lemma 6.3.3 we know that $\mathbb{D}$ is of the form $\bigcup_{i=1}^m \mathbb{E}_i \cap \mathbb{S}_i$, where $\mathbb{E}_i$ is definable in $\mathcal{R}$ and $\mathbb{S}_i$ is dense in $\mathcal{G}^n$. Now define

$$\mathbb{F}_i := (\bigcup_{i \neq j} \mathbb{E}_j) - \mathbb{E}_i, \mathbb{E} := \bigcup_{i=1}^m \mathbb{E}_i \text{ and } \mathbb{S} := \bigcup_{i=1}^m (\mathbb{S}_i - \mathbb{F}_i).$$

Since the intersection of the $\mathbb{F}_i$ is empty, we get that $\mathbb{S}$ is dense in $\mathcal{G}^n$ by Lemma 6.3.4. Note that $\mathbb{D} = \mathbb{E} \cap \mathbb{S}$.

Finally we just need consider the case of $n = 1$. By Lemma 6.3.3 we know that $\mathbb{D}$ is of the form $\bigcup_{i=1}^m \mathbb{E}_i \cap \mathbb{S}_i$, where $\mathbb{E}_i$ is definable in $\mathcal{R}$ and $\mathbb{S}_i$ is of the form $\gamma_i \mathcal{G}^{[d]}$, where $\gamma_i \in \Gamma$. Since $\mathfrak{L}_\Gamma^\tau$ has a constant for each $\gamma \in \Gamma$, the $\mathbb{S}_i$ are 0-definable. Hence all the conditions of Theorem 6.3.1 are satisfied and hence $(\tilde{\mathbb{R}}, \Gamma)$ has o-minimal open core.

□

Chapter 7

Discrete group and irrational powers

In this chapter we discuss the model theory of the structure $(\tilde{\mathbb{R}}, 2^{\mathbb{Z}})$. Since for τ not algebraic of order 2 the subgroup $2^{\mathbb{Z}}$ is definable in $(\tilde{\mathbb{R}}, 2^{\mathbb{Z}}2^{\mathbb{Z}\tau})$ by

$$\{x \in 2^{\mathbb{Z}}2^{\mathbb{Z}\tau} \mid x^\tau \in 2^{\mathbb{Z}}2^{\mathbb{Z}\tau}\},$$

this is the same as analyzing the structure consisting of $\tilde{\mathbb{R}}$ and the dense subgroup $2^{\mathbb{Z}}2^{\mathbb{Z}\tau}$.

These structures have a strong connection to other analytic structures. Miller and Speissegger showed in [MS06] that these occur naturally in the study of expansions of $\mathbb{R}_{an}$ by trajectories of analytic planar vector fields.

In Section 1 of this chapter we will show that $(\tilde{\mathbb{R}}, 2^{\mathbb{Z}})$ is not model complete. The rest of the chapter will then consider the connection between this structure and Diophantine approximation of τ.

7.1 $(\tilde{\mathbb{R}}, 2^{\mathbb{Z}})$ is not model complete

In this section an argument will be presented that shows that $(\tilde{\mathbb{R}}, 2^{\mathbb{Z}})$ is not model complete. It is not known whether there is any 'small' expansions of the language which would make the structure model complete. The following is based on the argument used by Marker in [M06] to show that the complex field together with exponentiation is not model complete.

Therefore note that an F_σ*-set* is a countable union of closed sets. A countable intersection of open sets is called a G_δ*-set*. The following lemma is a well-known corollary from the fact that the image of a compact set under a continuous function is compact.

Lemma 7.1.1 *Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a continuous function. Then the image of a closed set under f is an F_σ-set.*

The main tool in this section is the Baire Category Theorem.

Theorem 7.1.2 *Every intersection of countably many dense open sets of a non-empty complete metric space is dense.*

For more details, see for example [R73].

Corollary 7.1.3 *Let $D \subset \mathbb{R}_{>0}$ be dense and countable. Then D is* not *a G_δ-set.*

Proof: Suppose for a contradiction that D is a intersection of countably many open $A_i \subset \mathbb{R}_{>0}$. Since D is dense in $\mathbb{R}_{>0}$, every A_i is dense in $\mathbb{R}_{>0}$ as well. Further note that $\mathbb{R}_{\geq 0} - D$ is a G_δ-set, because

$$\mathbb{R}_{\geq 0} - D = \bigcap_{d \in D} \mathbb{R}_{\geq 0} - \{d\}.$$

Now we have that

$$\emptyset = \bigcap_{d \in D} \mathbb{R}_{\geq 0} - \{d\} \cap \bigcap_{i \in \mathbb{N}} A_i$$

Thus there is a countable intersection of dense open sets which intersection is empty. This is a contradiction to Theorem 7.1.2.

□

Again let $\mathfrak{L}^\tau$ be the language of $\tilde{\mathbb{R}}$ and let $\mathfrak{L}^\tau(G)$ be the language $\mathfrak{L}^\tau$ augmented by a unary predicate G.

Theorem 7.1.4 $(\tilde{\mathbb{R}}, 2^{\mathbb{Z}})$ *is not model complete.*

Proof: Suppose for a contradiction that $(\tilde{\mathbb{R}}, 2^{\mathbb{Z}})$ is model complete. Let $A \subset \mathbb{R}$ be definable in $(\tilde{\mathbb{R}}, 2^{\mathbb{Z}})$. First we want to show that A is a projection of a closed set. Therefore note that model completeness of $(\tilde{\mathbb{R}}, 2^{\mathbb{Z}})$ would imply that A is defined by formula of the form

$$\exists y_1 ... \exists y_n \varphi(z, y_1, ..., y_n),$$

where φ is a boolean combination of atomic $\mathfrak{L}^\tau(G)$-formulas. Now we can replace occurrences of $\neg G(x)$ in φ by

$$\exists y \; G(y) \wedge y < x < 2y.$$

Further $x \neq 0$ can be replaced by $\exists y \; x \cdot y = 1$ and $x \leq y$ by $\exists z \; x + z^2 = y$. Hence by modifying φ and increasing the number n of existential quantifiers, we can assume that the set $\{(a, \vec{b}) \in \mathbb{R}^{1+n} | (\tilde{\mathbb{R}}, 2^{\mathbb{Z}}) \models \varphi(a, \vec{b})\}$ is closed. Hence A is a projection of a closed set.

We will now show that this gives us a contradiction. Because the projection is continuous, we get by Lemma 7.1.1 that A is a F_σ-set. Further note that $\mathbb{R} - A$ is definable as well and hence it is an F_σ-set as well. Since the complement of a F_σ-set is a G_δ-set, A is also a G_δ-set. Now note that $2^{\mathbb{Z}} \cdot 2^{\tau \cdot \mathbb{Z}}$ is dense in $\mathbb{R}_{>0}$ and further definable in $(\tilde{\mathbb{R}}, 2^{\mathbb{Z}})$. Hence $2^{\mathbb{Z}} \cdot 2^{\tau \cdot \mathbb{Z}}$ is a countable, dense G_δ-set. This is a contradiction to Corollary 7.1.3.

□

7.2 λ and μ-arithmetic

In this section, we will consider two $(\tilde{\mathbb{R}}, 2^{\mathbb{Z}})$-definable functions $\lambda : \mathbb{R}_{>0} \to 2^{\mathbb{Z}}$ and $\mu : 2^{\mathbb{Z}} \to 2^{\mathbb{Z}} 2^{\tau\mathbb{Z}} \cap [1,2]$. We will show some easy properties of these functions which will be used in the next section.

Definition 7.2.1 *For a given positive real number c, we define $\lambda(c)$ to be the unique element of $2^{\mathbb{Z}}$ such that*

$$\lambda(c) \leq c < 2\lambda(c).$$

Further for every $g \in 2^{\mathbb{Z}}$, we set

$$\mu(g) := \frac{g^\tau}{\lambda(g^\tau)}.$$

First note that $\lambda(g) = g$ if and only if $g \in 2^{\mathbb{Z}}$. Further note that $\mu(g) \in [1,2]$. Also note that

$$2^{-1}\lambda(g^\tau)^{-1} < g^{-\tau} < \lambda(g^\tau)^{-1}$$

Hence $\mu(g^{-1}) = 2\mu(g)^{-1}$.

Proposition 7.2.2 *Let $g, h \in 2^{\mathbb{Z}}$. Then*

$$\mu(gh) = \begin{cases} \mu(g) \cdot \mu(h), & \textit{if } \mu(g) \cdot \mu(h) < 2; \\ \mu(g) \cdot \mu(h) \cdot 2^{-1}, & \textit{otherwise.} \end{cases}$$

Proof: By definition, we have that $\lambda(g^\tau) < g^\tau < 2\lambda(g^\tau)$ and $\lambda(h^\tau) < h^\tau < 2\lambda(h^\tau)$. Hence

$$\lambda(g^\tau) \cdot \lambda(h^\tau) < (gh)^\tau < 4 \cdot \lambda(g^\tau) \cdot \lambda(h^\tau).$$

Thus

$$\lambda((gh)^\tau) = \begin{cases} \lambda(g^\tau) \cdot \lambda(h^\tau), & \text{if } (gh)^\tau < 2 \cdot \lambda(g^\tau) \cdot \lambda(h^\tau); \\ 2 \cdot \lambda(g^\tau) \cdot \lambda(h^\tau), & \text{otherwise.} \end{cases}$$

□

Corollary 7.2.3 *Let $g, h \in 2^{\mathbb{Z}}$. Then*

$$\mu(gh^{-1}) = \begin{cases} \mu(g) \cdot \mu(h)^{-1}, & \textit{if } \mu(g) > \mu(h); \\ \mu(g) \cdot \mu(h)^{-1} \cdot 2, & \textit{otherwise.} \end{cases}$$

Proof: Proposition 7.2.2 implies that

$$\mu(gh^{-1}) = \begin{cases} \mu(g) \cdot \mu(h)^{-1}, & \text{if } \mu(g)\mu(h^{-1}) < 2; \\ \mu(g) \cdot \mu(h)^{-1} \cdot 2^{-1}, & \text{otherwise.} \end{cases}$$

Note that by the remark before Proposition 7.2.2, we have that

$$\mu(g)\mu(h^{-1}) = \mu(g) \cdot \mu(h)^{-1} \cdot 2.$$

Hence $\mu(g)\mu(h^{-1}) < 2$ if and only if $\mu(g) < \mu(h)$.

□

In the earlier results on dense multiplicative subgroups, it was critical that certain subgroups like the subgroup of n-th powers are dense as well. In the following, we note some similar results on the density of $\mu(A)$ in $(1,2)$, where A is some subset of $2^{\mathbb{Z}}$. Therefore for $n \in \mathbb{N}$, let $P_n(x)$ be the $\mathfrak{L}^{\tau}(G)$-formula defined by

$$G(x) \wedge \exists y\ G(y) \wedge y^n = x.$$

Lemma 7.2.4 *Let $n \in \mathbb{N}$ and let $c, d \in (1,2)$ with $c < d$. Then there is a $g \in 2^{\mathbb{Z}}$ such that $c < \mu(g) < d$ and $P_n(g)$.*

Proof: By the density of $\mu(2^{\mathbb{Z}})$ in $(1,2)$, there is $h \in 2^{\mathbb{Z}}$ such that $1 < c^{\frac{1}{n}} < \mu(h) < d^{\frac{1}{n}} < 2$. Then by Proposition, 7.2.2 $c < \mu(h)^n < d$.

□

Proposition 7.2.5 *Let $l \in \mathbb{N}$. Let $n_i, m_i \in \mathbb{N}$ and $h_i \in 2^{\mathbb{Z}}$ for every $i = 1, ..., l$ and let $c, d \in (1,2)$ with $c < d$. If there is $g \in 2^{\mathbb{Z}}$ with $P_{n_i}(g^{m_i} h_i)$ for every $i = 1, ..., l$, then there is a $g \in 2^{\mathbb{Z}}$ such that $c < \mu(g) < d$ and $P_{n_i}(g^{m_i} h_i)$ for every $i = 1, ..., l$.*

Proof: Let $g_0 \in 2^{\mathbb{Z}}$ satisfy $P_{n_i}(g_0^{m_i} h_i)$ for all $i = 1, ..., l$. If $c < \mu(g_0) < d$, the proof is complete. So assume that this is not the case. Take $g_1 \in 2^{\mathbb{Z}}$ such that $P_{\prod_{i=1}^{l} n_i}(g_1)$ and

$$\begin{aligned} \frac{c}{\mu(g_0)} < \mu(g_1) < \frac{d}{\mu(g_0)} & \qquad \text{if } c > \mu(g_0) \\ \frac{2c}{\mu(g_0)} < \mu(g_1) < \frac{2d}{\mu(g_0)} & \qquad \text{if } d < \mu(g_0). \end{aligned}$$

Now for $i = 1, ..., l$, we get that $P_{n_i}((g_1 \cdot g_0)^{m_i} h_i)$. First suppose $c > \mu(g_0)$. Since $\mu(g_1) \cdot \mu(g_0) < d < 2$, we have

$$c < \mu(g_1)\mu(g_0) = \mu(g_1 \cdot g_0) < d.$$

Now suppose $c < d < \mu(g_0)$. Then $\mu(g_1) \cdot \mu(g_0) > 2c > 2$. Hence

$$c < \mu(g_1) \cdot \mu(g_0) \cdot 2^{-1} = \mu(g_1 \cdot g_0) < d.$$

□

7.3 Connection to Diophantine approximation

In this section, we explore the connection between $(\tilde{\mathbb{R}}, 2^{\mathbb{Z}})$ and Diophantine approximation. We will use results from Diophantine approximation in order to analyze how definable sets depend on τ. We will see that statements on the Diophantine approximation of τ are expressible in this structure. First results on the structure of such definable subsets of $(\tilde{\mathbb{R}}, 2^{\mathbb{Z}})$ will be given. Unfortunately, we can not answer the question yet whether $\mathbb{Z}$ is definable in $(\tilde{\mathbb{R}}, 2^{\mathbb{Z}})$. We will construct a small, dense and co-dense definable subset S of $(1, 2)$ for which it is not known whether there are any $\mathbf{cl}_T$-dependency between S and $2^{\mathbb{Z}}$. This set S will be $\forall\exists$-definable. Such a set is a candidate to be a counter example to the near model completeness of $(\tilde{\mathbb{R}}, 2^{\mathbb{Z}})$. In the last part of this section we will analyze certain $\forall\exists\forall$-definable sets in $(\tilde{\mathbb{R}}, 2^{\mathbb{Z}})$. It will be shown that under certain assumptions on τ, this sets are already definable in $\tilde{\mathbb{R}}$. Further analysis along these lines may lead to better results on the complexity of definable subsets of $(\tilde{\mathbb{R}}, 2^{\mathbb{Z}})$.
Note that everything proved in this section also holds, if one replaces $\tilde{\mathbb{R}}$ by any other polynomially bounded expansion of the real field with field of exponents $\mathbb{Q}(\tau)$.

7.3.1 Approximable numbers

First we consider when a real number τ is approximable by decreasing functions.

Definition 7.3.1 *Let* $\psi : \mathbb{N} \to \mathbb{R}_{>0}$ *with* $\lim_{x\to\infty} \psi(x) = 0$. *We define the* set of ψ-approximable numbers $W(\psi)$ *by*

$$\{\alpha \in (0, 1) \mid |p\alpha - q| < \psi(q), \text{ for infinitely many } p \in \mathbb{Z}, q \in \mathbb{N}\}.$$

The following is Khinchin's Theorem on Diophantine Approximation. A proof can be found in [BD99].

Theorem 7.3.2 *The Lebesgue measure of* $W(\psi)$ *is given by*

$$|W(\psi)| = \begin{cases} 0, & \text{if } \sum_{i=1}^{\infty} \psi(i) < \infty; \\ 1, & \text{if } \sum_{i=1}^{\infty} \psi(i) = \infty \text{ and } \psi \text{ is non-increasing.} \end{cases}$$

Corollary 7.3.3 *The Lebesgue measure of* $\mathcal{T} := (0, 1) - \bigcup_{d\in\mathbb{N},d\geq 2} W(x \mapsto x^{-d})$ *is* 1.

Proof: By Theorem 7.3.2 and the fact that for every $d > 1$ the sum $\sum_{i=1}^{\infty} i^d$ converges, we get that the Lebesgue measure of $W(x \mapsto x^{-d})$ is 0 for every $d \in \mathbb{N}_{>1}$. Because the countable union of null sets is a null set, the Lebesgue measure of $\mathcal{T}$ is 1.

□

Proposition 7.3.4 *Let $d \in \mathbb{N}_{>1}$ and $f : \mathbb{R} \to \mathbb{R}$ be a definable function in $\tilde{\mathbb{R}}$ which converges to 0. Then*

$$\log_2(1 + f(2^n)) < \frac{1}{n^d}$$

for large enough $n \in \mathbb{N}$.

Proof: Note that

$$\log_2(1 + f(2^n)) < \frac{1}{n^d} \text{ iff } 1 + f(2^n) < 2^{\frac{1}{n^d}}.$$

By polynomial boundedness of $\tilde{\mathbb{R}}$, there is $p \in \mathbb{Q}(\tau)$ with $p < 0$ and $c \in \mathbb{R}$ such that

$$f(2^n) = c \cdot 2^{pn} + o(2^{pn}).$$

Hence

$$\begin{aligned} 1 + f(2^n) < 2^{\frac{1}{n^d}} &\text{ iff } 1 + c \cdot 2^{pn} + o(2^{pn}) < 2^{\frac{1}{n^d}} \\ &\text{ iff } c + \frac{o(2^{pn})}{2^{pn}} < \frac{2^{\frac{1}{n^d}} - 1}{2^{pn}}. \end{aligned}$$

The left hand side converges to c, while by the L'Hospital rule

$$\begin{aligned} \lim_{n\to\infty} \frac{2^{\frac{1}{n^d}} - 1}{2^{pn}} &= \lim_{n\to\infty} \frac{\ln(2) \cdot (-d) \cdot n^{-(d+1)} \ln(2) 2^{\frac{1}{n^d}}}{p 2^{pn}} \\ &= \lim_{n\to\infty} (-d) \cdot n^{-(d+1)} \cdot p^{-1} \cdot 2^{-pn+n^{-2}} \\ &= \infty, \text{ since } p < 0. \end{aligned}$$

□

Corollary 7.3.5 *Let $\tau \in \mathcal{T}$, and let $f : \mathbb{R} \to \mathbb{R}$ be a definable function in $\tilde{\mathbb{R}}$ which converges to 0. Then there are only finitely many $g \in 2^{\mathbb{Z}}$ such that $g > 1$ and*

$$1 < \mu(g) < 1 + f(g). \tag{7.3.1}$$

Proof: Let $d \in \mathbb{N}_{>2}$ with $\tau \in W(x \mapsto x^d)$. Suppose for a contradiction that there are infinitely many $m \in \mathbb{N}$ such that $1 < \mu(2^m) < 1 + f(2^m)$. For such an m, let $l \in \mathbb{Z}$ be such that $2^l = \lambda(2^{\tau \cdot m})$. This implies that

$$1 < \mu(2^m) = \frac{2^{\tau \cdot m}}{2^l} = 2^{\tau \cdot m - l} < 1 + f(2^m).$$

Thus

$$0 < \tau \cdot m - l < log_2(1 + f(2^m)).$$

By Proposition 7.3.4, this implies that for all, but finitely such m

$$\tau \cdot m - l < (1/m)^d. \tag{7.3.2}$$

But since $\tau \in W(x \mapsto x^d)$, there are only finitely m satisfying inequality (7.3.2). This contradicts the assumption that there are infinitely many 2^m satisfying (7.3.1).

□

Corollary 7.3.6 *Let $\tau \in \mathcal{T}$. Let $f : \mathbb{R} \to \mathbb{R}$ be definable in $\tilde{\mathbb{R}}$ with $\lim_{x\to\infty} f(x) = 1$ and let $c \in \mu(2^{\mathbb{Z}})$. Then there are finitely many $g \in 2^{\mathbb{Z}}$ such that $g > 1$ and*

$$c < \mu(g) < c \cdot f(g).$$

Proof: By o-minimality of $\tilde{\mathbb{R}}$ and Theorem 2.1.2, there is $b \in \mathbb{R}$ such that $f|_{(b,\infty)}$ is strictly decreasing. Without loss of generality we can assume that $f(x) = f(b)$ for all $0 < x < b$. Hence f is decreasing. By Corollary 7.3.5, there is $n \in \mathbb{N}$ such that there are at most n many $g \in 2^{\mathbb{Z}}$ such that $g > 1$

$$1 < \mu(g) < f(g).$$

Let $l_1, ..., l_n$ be these elements of $2^{\mathbb{Z}}$. Now let $c = \mu(h)$ for some $h \in 2^{\mathbb{Z}}$ and $h > 1$. Suppose there is $g \in 2^{\mathbb{Z}}$ with $c < \mu(g) < c \cdot f(g)$. This implies that $\frac{\mu(g)}{\mu(h)} < f(g)$. Note that $\mu(g) > \mu(h)$ and hence $\mu(gh^{-1}) = \frac{\mu(g)}{\mu(h)}$ by Corollary 7.2.3. Since f is decreasing, we get that

$$\mu(gh^{-1}) < f(g) \leq f(gh^{-1}). \tag{7.3.3}$$

Hence there are at most n $g > h$ satisfying (7.3.3) which are exactly $h \cdot l_1, ..., h \cdot l_n$.

□

But Corollary 7.3.6 does not hold for arbitrary $c \in \mathbb{R}$.

Proposition 7.3.7 *Let $\tau \in \mathbb{R}$ arbitrary. Let $f : \mathbb{R} \to \mathbb{R}$ be definable in $\tilde{\mathbb{R}}$ with $\lim_{x\to\infty} f(x) = 1$, and let $a, b \in (1, 2)$ be real numbers with $a < b$. Then there is a real number $c \in (a, b)$ such that there are infinitely many $g \in 2^{\mathbb{Z}}$ with $g > 1$ and*

$$c < \mu(g) < c \cdot f(g).$$

Proof: In order to construct such a real number c, we will define recursively an increasing sequence $(g_n)_{n\in\mathbb{N}}$ of elements of $2^{\mathbb{Z}}$ as follows. Let $g_0 \in 2^{\mathbb{Z}}$ such that on (g_0, ∞) the function f is decreasing, larger than 1 and smaller than $\frac{b}{a}$. Now by density of $\mu(2^{\mathbb{Z}})$ in $(1,2)$, there is $g_1 \in 2^{\mathbb{Z}}$ such that

$$af(g_0) < \mu(g_1) < b.$$

Since f is decreasing and larger than 1, we also have

$$a < \frac{\mu(g_1)}{f(g_1)} < \mu(g_1).$$

Suppose $g_1 < ... < g_n$ are already defined with

$$a < \frac{\mu(g_1)}{f(g_1)} < ... < \frac{\mu(g_n)}{f(g_n)} < \mu(g_n) < ... < \mu(g_1) < b.$$

Let $g_{n+1} \in 2^{\mathbb{Z}}$ be such that $g_{n+1} > g_n$

$$\frac{f(2g_n)}{f(g_n)} \cdot \mu(g_n) < \mu(g_{n+1}) < \mu(g_n).$$

Since f is decreasing, this implies that

$$\frac{\mu(g_n)}{f(g_n)} < \frac{\mu(g_{n+1})}{f(2g_n)} < \frac{\mu(g_{n+1})}{f(g_{n+1})} < \mu(g_{n+1}) < \mu(g_n).$$

Thus, we have defined a sequence $(g_n \in 2^{\mathbb{Z}})_{n\in\mathbb{N}}$ such that

$$g_i > g_j, \mu(g_i) < \mu(g_j) \text{ and } \frac{\mu(g_i)}{f(g_i)} > \frac{\mu(g_j)}{f(g_j)}, \text{ for every } i > j.$$

Set $c := \lim_{n\to\infty} \mu(g_n)$. Note that

$$\lim_{n\to\infty} \frac{\mu(g_n)}{f(g_n)} = \frac{\lim_{n\to\infty} \mu(g_n)}{\lim_{n\to\infty} f(g_n)} = c.$$

Thus for every $n \in \mathbb{N}$, we have that

$$a < \frac{\mu(g_n)}{f(g_n)} < c < \mu(g_n) < b.$$

□

The following Corollary follows directly from Corollary 7.3.6 and Proposition 7.3.7.

Corollary 7.3.8 *Let $\tau \in \mathcal{T}$. Let $f : \mathbb{R} \to \mathbb{R}$ be definable in $\tilde{\mathbb{R}}$ with $\lim_{x\to\infty} f(x) = 1$. Let $S_f \subseteq \mathbb{R}$ be the $\mathfrak{L}^\tau(G)$-definable set*

$$\{c \in (1,2) \mid \forall g \in 2^{\mathbb{Z}} \exists h \in 2^{\mathbb{Z}}\ h > g \wedge c < \mu(h) < c \cdot f(h)\}.$$

Then S_f is dense and co-dense in $(1,2)$ and $S_f \cap \mu(2^{\mathbb{Z}})$ is empty.

Note that by definition S_f is $\forall\exists$-definable. It is not obvious how the set S_f looks like. Especially, it is not clear whether there is any $\mathbf{cl}_T$-dependency between element of S_f and $2^{\mathbb{Z}}$. The only thing, one can prove is that S is small.

Proposition 7.3.9 *Let $\tau \in \mathcal{T}$. Let $f : \mathbb{R} \to \mathbb{R}$ be definable in $\tilde{\mathbb{R}}$ with $\lim_{x\to\infty} f(x) = 1$. Then $S_f \subseteq \mathbb{R}$ has Lebesgue measure 0.*

This directly implies that $S_f \neq (1,2) - \mu(2^{\mathbb{Z}})$. For the proof of Proposition 7.3.9, we need the Borel-Cantelli Lemma. See [B04] Lemma 1.3 for a proof.

Lemma 7.3.10 *Let $(E_n)_{n\in\mathbb{N}}$ be a sequence of Lebesgue measurable subsets of $\mathbb{R}$ such that $\sum_{n\geq 0} P(E_n)$ converges, where $P(E_n)$ is the Lebesgue measure of E_n. Then the Lebesgue measure of*

$$\bigcap_{N\geq 1} \bigcup_{n\geq N} E_n$$

is 0.

<u>Proof of Proposition 7.3.9:</u> By definition S_f is equal to

$$\bigcap_{N\geq 1} \bigcup_{n\geq N} \left(\frac{\mu(2^n)}{f(2^n)}, \mu(2^n)\right).$$

For every $n \in \mathbb{N}$ the Lebesgue measure of $(\frac{\mu(2^n)}{f(2^n)}, \mu(2^n))$ is

$$\mu(2^n) - \frac{\mu(2^n)}{f(2^n)}.$$

Since $\mu(2^n) < 2$, the Lebesgue measure is smaller than $2 \cdot (1 - \frac{1}{f(2^n)})$. By the Borel-Cantelli Lemma 7.3.10, it just left to show that for large enough $n \in \mathbb{N}$

$$\left(1 - \frac{1}{f(2^n)}\right) < \frac{1}{n^2}. \tag{7.3.4}$$

Let $g : \mathbb{R} \to \mathbb{R}$ be the function defined by $g(x) = (1 - \frac{1}{f(x)})$. Note that g is definable in $\tilde{\mathbb{R}}$ and $\lim_{x\to\infty} g(x) = 0$. By polynomial-boundedness of $\tilde{\mathbb{R}}$, there is a $p \in \mathbb{Q}(\tau)$ and $c \in \mathbb{R}$ such that $p < 0$ and

$$g(x) = c \cdot x^p + o(x^p).$$

This implies that

$$\lim_{n\to\infty} g(2^n) \cdot n^2 = \lim_{n\to\infty} (c \cdot 2^{np} + o(2^{np})) \cdot n^2 = 0$$

Hence inequality (7.3.4) holds for large enough n.

□

7.3.2 Best approximations

In this section, we consider ∀∃∀-definable sets. In particular, we analyze formulas of the form

$$\forall k \in G \exists g \in G \; k < g \wedge \forall h \in G \; \varphi(g, \mu(g), h, \mu(h)),$$

where φ is a $\mathfrak{L}^{\tau}$-formula. While the formula

$$\forall k \in G \exists g \in G \; k < g \wedge \forall h \in G \; (h < g) \rightarrow (\min\{\mu(g)-1, 2-\mu(g)\} < \min\{\mu(h)-1, 2-\mu(h)\})$$

surely holds in $(\tilde{\mathbb{R}}, 2^{\mathbb{Z}})$, it depends heavily on τ again whether or not for a given function $f : \mathbb{R} \to \mathbb{R}$ definable in $\tilde{\mathbb{R}}$ the following formula φ_f holds in $(\tilde{\mathbb{R}}, 2^{\mathbb{Z}})$:

$$\begin{aligned} \varphi_f := {} & \forall k \in G \exists g \in G \; k < g \wedge \forall h \in G \\ & (h < f(g) \wedge h \neq g) \rightarrow (\min\{\mu(g) - 1, 2 - \mu(g)\} < \min\{\mu(h) - 1, 2 - \mu(h)\}) \end{aligned}$$

We fix some further notation from the theory of Diophantine approximation.

Definition 7.3.11 *Let $n \in \mathbb{N}$. We say n is a* best rational approximation to τ *if there is $d \in \mathbb{N}$ such that for all $m, c \in \mathbb{N}$ with $m < n$*

$$|n \cdot \tau - d| < |m \cdot \tau - c|.$$

Let $(n_{\tau,i})_{i \in \mathbb{N}}$ be the series of best rational approximations to τ. We set

$$L_\tau := \limsup_{i \to \infty} \frac{n_{\tau,i+1}}{n_{\tau,i}}.$$

It directly follows from the definition of a best rational approximation that

Lemma 7.3.12 *Let $n \in \mathbb{N}$. Then n is a best rational approximation to τ iff for all $h \in 2^{\mathbb{Z}}$*

$$(1 < h < 2^n) \rightarrow (\min\{\mu(2^n) - 1, 2 - \mu(2^n)\} < \min\{\mu(h) - 1, 2 - \mu(h)\})$$

Theorem 7.3.13 *Let $\tau \in \mathbb{R}$. Let $f : \mathbb{R} \to \mathbb{R}$ be definable in $\tilde{\mathbb{R}}$, and let $c \in \mathbb{R}$ with $\lim_{x \to \infty} \frac{f(x)}{cx^q} = 1$. If $q < L_\tau$, then*

$$(\tilde{\mathbb{R}}, 2^{\mathbb{Z}}) \models \varphi_f.$$

If $q > L_\tau$, then

$$(\tilde{\mathbb{R}}, 2^{\mathbb{Z}}) \models \neg\varphi_f.$$

Proof: We just consider the case of $q < L_\tau$. The case of $q > L_\tau$ can be shown similarly. Let $N \in \mathbb{N}$ be so large such that for all $i > N$

$$q < q' < \frac{n_{\tau,i+1}}{n_{\tau,i}} \tag{7.3.5}$$

and for all $x \in \mathbb{R}$ with $x > 2^N$

$$f(x) < x^{q'}.$$

Let $g = 2^n \in 2^{\mathbb{Z}}$, where n is a best rational approximation to τ. Suppose there is $h \in 2^{\mathbb{Z}}$ with $1 < h < f(g)$ and

$$\min\{\mu(g) - 1, 2 - \mu(g)\} < \min\{\mu(h) - 1, 2 - \mu(h)\}.$$

Since n is a best approximation, we have $g < h < f(g)$ by Lemma 7.3.12. Further since $f(g) < g^{q'}$, we know that $h < g^{q'}$. Now let h be minimal above g and $m \in \mathbb{N}$ with $2^m = h$. This implies that m itself is a best approximation. Because n and m are consecutive best rational approximation of τ and $n > N$ and $m > N$, we have that $q' \cdot n < m$ by (7.3.5). This is implies that $g^{q'} < h$, which is a contradiction.

□

By Theorem 2.1.14, we get the following quantifier elimination result.

Corollary 7.3.14 *Let $\tau \in \mathbb{R}$ and $L_\tau \notin \mathbb{Q}(\tau)$. Further let $f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ be a 0-definable function in $(\tilde{\mathbb{R}}, 2^{\mathbb{Z}})$. Then the set*

$$\{\vec{a} \in \mathbb{R}^n \mid (\tilde{\mathbb{R}}, 2^{\mathbb{Z}}) \models \varphi_{f(\vec{a})(-)}\}$$

is already 0-definable in $\tilde{\mathbb{R}}$.

Note that it is not known whether the condition $L_\tau \notin \mathbb{Q}(\tau)$ can be dropped from the previous corollary. It is well known that the set of best rational approximations of a irrational number τ is determined by its continued fraction. For an introduction to continued fractions see for example [L66], [B04] or [G06]. The next theorem is Theorem 6 of Chapter 1 in [L66].

Theorem 7.3.15 *Let $\tau \in \mathbb{R} - \mathbb{Q}$ and let $[a_0; a_1, a_2, ...]$ be the continued fraction expansion of τ. Then the best rational approximations are given recursively by*

$$n_{\tau,0} = 1, n_{\tau,1} = a_1 \text{ and } n_{\tau,i} = a_i n_{\tau,i-1} + n_{\tau,i-2}$$

This theorem implies that one example for a real number τ with $L_\tau \in \mathbb{Q}(\tau)$ is $\phi := \frac{1+\sqrt{5}}{2}$, the golden ratio. The continued fraction expansion of ϕ is just $[1; 1, 1, ...]$. Hence by Theorem 7.3.15 the set of best approximations to ϕ are the Fibonacci numbers. Since the ratio between two consecutive Fibonacci numbers converges to ϕ, we get that $L_\phi = \phi$. But again such behavior is the exception and not the rule.

Theorem 7.3.16 *The set $\mathcal{V}$ of numbers τ with $L_\tau < \infty$ has Lebesgue measure 0.*

Proof: If $[a_0; a_1, a_2, ...]$ is the continued fraction expansion of a real number τ, we say that $(a_i)_{i\in\mathbb{N}}$ is sequence of partial quotients of τ. By Theorem 1.9 and Corollary 1.6 of [B04], the set of numbers whose sequences of partial quotients is bounded, has Lebesgue measure 0. By Theorem 7.3.15, the sequence of partial quotients of real number τ is bounded if and only if $L_\tau < \infty$.

□

Note that all real quadratic number are in $\mathcal{V}$. Examples for such real numbers which are not in $\mathcal{V}$ are Euler's number e and $\tan(1)$.

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