

# The Width of Verbal Subgroups in Profinite Groups



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Dedicated to my parents



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# Abstract

The main result of this thesis is an original proof that every word has finite width in a compact  $p$ -adic analytic group. The proof we give here is an alternative to Andrei Jaikin-Zapirain's recent proof of the same result, and utilises entirely group-theoretical ideas. We accomplish this by reducing the problem to a proof that every word has finite width in a profinite group which is virtually a polycyclic pro- $p$  group. To obtain this latter result we first establish that such a group can be embedded as an open subgroup of a group of the form  $N_1M_1$ , where  $N_1$  is a finitely generated closed normal nilpotent subgroup, and  $M_1$  is a finitely generated closed nilpotent-by-finite subgroup; we then adapt a method of V. A. Roman'kov. As a corollary we note that our approach also proves that every word has finite width in a polycyclic-by-finite group (which is not profinite).

As a supplementary result we show that for finitely generated closed subgroups  $H$  and  $K$  of a profinite group the commutator subgroup  $[H, K]$  is closed, and give examples to show that various hypotheses are necessary. This implies that the outer-commutator words have finite width in profinite groups of finite rank. We go on to establish some bounds for this width.

In addition, we show that every word has finite width in a product of a nilpotent group of finite rank and a virtually nilpotent group of finite rank. We consider the possible application of this to soluble minimax groups.



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# Introduction

*In the beginning was the Word* (John 1:1)

The canonical example of an abstract word is the commutator word  $[x_1, x_2] := x_1^{-1}x_2^{-1}x_1x_2$ . The derived subgroup of a group  $G$  is then the subgroup of  $G$  generated by all the elements of the form  $[g_1, g_2]$ , for  $g_1, g_2 \in G$ . Given an arbitrary word  $w$ , a generalisation of this idea is to consider the subgroup generated by all the elements of  $G$  which can be expressed in the form of  $w$ . Such a subgroup is called a verbal subgroup. Common verbal subgroups include the terms of the derived series and the lower central series.

A significant use of verbal subgroups is to form the smallest subgroup modulo which certain laws hold. In this way verbal subgroups serve a useful rôle in the study of varieties; for any group which is free in a variety is isomorphic to the quotient of a free group by a verbal subgroup. Furthermore, in any finitely generated group every subgroup of finite index contains a verbal subgroup of finite index; so verbal subgroups are in abundance.

The width of a verbal subgroup  $w(G)$  is defined to be the smallest number  $\|w\|$  such that every element of  $w(G)$  can be written as a product of  $\|w\|$  values of  $w$  or their inverses. When the width of  $w(G)$  is finite we say that  $w$

has finite width in  $G$ . A group in which every word has finite width is said to be *verbally-elliptic*. While these properties are interesting in their own right, they have particular significance when dealing with profinite groups, for a verbal subgroup of a profinite group is *closed* exactly when it has finite width. The derived group of any finitely generated profinite group is closed, but Roman'kov [29] has provided an example of a finitely generated *pro-p* group in which the second derived group is not closed; therefore the study of the verbal subgroups of profinite groups is anything but trivial. Despite this, a good deal of progress has been made in this subject area during the last few years: Andrei Jaikin-Zapirain [14] has shown that compact  $p$ -adic analytic groups are verbally-elliptic, and has deduced a structural property which describes exactly when a word has finite width in all finitely generated *pro-p* groups. Here we provide an alternative proof for the former result which is more elementary and uses entirely group-theoretical ideas. We also consider some specific words which do not have Jaikin-Zapirain's structural property. For some of these words we construct explicit groups in which their width is not finite, and for other words we show that, under certain additional assumptions, their verbal subgroup does indeed have finite width.

We now outline the contents in greater detail: **Chapter 1** is a concise introduction to the elementary properties of words, and their behaviour in profinite groups. Much of the terminology and notation regarding verbal subgroups is yet to become standard, so we introduce there the terminology which we will use throughout.

**Chapter 2** is a stand-alone account of commutators and commutator words. A word is said to be a commutator word if in every group the

corresponding verbal subgroup lies within the derived group. The outer-commutator words are those such as  $[[x_1, x_2, x_3], [x_4, x_5]]$ ; that is, built from commutators in such a way that every entry is a single variable, and all the variables are different.

We begin by describing a finitely generated soluble pro- $p$  group  $\mathfrak{R}$  in which the word  $[[x_1, x_2], [x_1, x_2, x_3]]$  has infinite width. The particular construction we use is based on a similar counterexample of Roman'kov [29]. We then extend a result of Nikolov and Segal [25] to show that *for finitely generated closed normal subgroups  $H$  and  $K$  of any profinite group, the subgroup  $[H, K]$ , generated by the elements  $[h, k]$  with  $h \in H$  and  $k \in K$ , is closed*. This then implies that *the width of an outer-commutator word is finite in a profinite group of finite rank*. Here the term finite rank is used to describe a profinite group in which every closed subgroup is finitely generated (topologically) by a bounded number of generators. The smallest possible such bound is then said to be the rank of the group. A key result is that *for a pro-nilpotent group of rank  $d$ , an outer-commutator word in  $n + 1$  variables has width no greater than  $(2d)^n$* . We also establish a bound of  $(144d^2 + 93d)^n$  in the case of pro-soluble groups.

**Chapter 3** is not about verbal subgroups. Instead it is intended to be a separate account describing a particular property of profinite groups  $G$  which contain a polycyclic pro- $p$  subgroup which is open and normal in  $G$ . Specifically, we show that  *$G$  can be embedded as an open subgroup of a profinite group  $N_1M_1$ , where  $N_1$  is a finitely generated closed normal nilpotent subgroup, and  $M_1$  is a finitely generated closed nilpotent-by-finite subgroup*. The character of this chapter is somewhat different to the others: The proof

which we shall give is neat, but it requires results from the cohomology of groups, and uses the Mal'cev completion of a finitely generated torsion-free nilpotent pro- $p$  group. Neither of these ideas are central topics of this thesis, but both require a large amount of theory to describe well; therefore we settle for stating the important properties which we use.

The result of Chapter 3 is integral to **Chapter 4**, where we show that the study of the width of words in compact  $p$ -adic analytic groups reduces to the study of the width of words in profinite groups which contain an open normal polycyclic pro- $p$  subgroup. We introduce Roman'kov's theory of generalised words, and provide an original exposition of their elementary properties. We then use these generalised words to show that the width of every word is finite in a profinite group of the form  $N_1M_1$ , where  $N_1$  is a closed normal nilpotent subgroup, and  $M_1$  is a closed virtually nilpotent subgroup. A technique of Roman'kov is then adapted to show that *every word has finite width in a profinite group which contains an open normal polycyclic pro- $p$  subgroup*. We show that this in turn implies that *every word has finite width in a compact  $p$ -adic analytic group*.

We end Chapter 4 with a brief consideration of soluble groups of finite rank. We show that *nilpotent-by-finite groups of finite rank are verbally-elliptic* and extend this result to soluble groups of finite rank which can be written as the product  $NM$  of a normal nilpotent subgroup  $N$  and a nilpotent-by-finite subgroup  $M$ .

I provide brief notes at the end of each chapter which are intended to describe my motivation for the material in the chapter, along with any peripheral ideas I may have.

# Index of notation

Throughout, the symbol ‘:=’ is used to mean *is defined to be*, as opposed to ‘=’ which is used to denote the relation *is equal to*. Numbers in brackets refer to the page number in the text where the original definition can be found.

$\mathbb{N}$	the natural numbers
$\mathbb{N}_0$	the natural numbers together with 0
$\mathbb{Z}$	the integers
$\mathbb{Z}_p$	the $p$ -adic integers
$\mathbb{Q}_p$	the $p$ -adic numbers
$\mathbb{F}_q$	the finite field of size $q$
$C_n$	the cyclic group of order $n$
$\overline{X}$	the closure of the set $X$
$A \subseteq B$	$A$ is a subset of $B$
$A \leq B$	$A$ is a subgroup of $B$
$A \leq_o B$	$A$ is an open subgroup of $B$
$A \leq_c B$	$A$ is a closed subgroup of $B$
$A \triangleleft B$	$A$ is a normal subgroup of $B$
$A \triangleleft_o B$	$A$ is an open normal subgroup of $B$
$A \triangleleft_c B$	$A$ is a closed normal subgroup of $B$
$A \times B$	the direct product of $A$ and $B$
$A \rtimes B$	the semidirect product of $A$ by $B$
$A \wr B$	the wreath product of $A$ by $B$
$A^{(n)}$	the direct product of $A$ with itself $n$ times
$d(G)$	the minimal number of generators of $G$
$\text{rk}(G)$	the rank of $G$ (14)
$\dim(G)$	the dimension of $G$ (47)

$\langle X \rangle$	the group generated by the set $X$
$X_n$	the group of words in $n$ variables (2)
$X_\infty$	the group of all words in any number of variables (2)
$F(X)$	the free group on the set $X$
$FG(A, \Psi)$	the free group for generalised words on the set $A$ (70)
$\text{Aut } G$	the automorphism group of $G$
$\text{Fitt}(G)$	the Fitting subgroup of $G$ (46)
$\tau(G)$	the maximal finite normal subgroup of $G$
$\mathbb{F}_q G$	the group algebra of $G$ over $\mathbb{F}_q$
$G^k$	$\langle g^k \mid g \in G \rangle$
$x^y$	$y^{-1}xy$
${}^y x$	$yx y^{-1}$
$[x, y]$	$x^{-1}y^{-1}xy$
$[x_1, \dots, x_n]$	$[[x_1, \dots, x_{n-1}], x_n]$
$[x, {}_n y]$	$[x, y, \dots, y]$ , with $n$ occurrences of $y$
$[H, K]$	$\langle [h, k] \mid h \in H, k \in K \rangle$ (19)
$[h, K]$	$\{[h, k] \mid k \in K\}$ (19)
$\mathfrak{C}\{H, K\}$	$\{[h, k] \mid h \in H, k \in K\}$ (31)
$G' = [G, G]$	the derived group of $G$ (19)
$w\{G\}$	the vocabulary of $w$ in $G$ (2)
$w(G)$	the verbal subgroup of $w$ in $G$ (2)
$w^*(G)$	the marginal subgroup of $w$ in $G$ (3)
$v \diamond w$	the independent product of words $v$ and $w$ (8)
$[v \mid w]$	the independent commutator of words $v$ and $w$ (21)
$\gamma_n, \gamma_n(G)$	the $n$ th lower central word (22), the $n$ th term in the lower central series for $G$
$\delta_n, \delta_n(G)$	the $n$ th derived word (22), the $n$ th derived group of $G$

$\ g\ _X, \ H\ _X$	the width of the element $g$ , or the subset $H$ , with respect to the set $X$ (5)
$\ w(G)\ , \ w\ $	the width of $w(G)$ with respect to $w\{G\}$ (6)
$X^{*k}$	the set of all length $k$ products in the elements of $X$ and their inverses (5)
$(\alpha_{ij})$	the matrix with entries $\alpha_{ij}$
$(A)_{ij}$	the $ij$ th entry of the matrix $A$
$A^T$	the transpose of the matrix $A$
$I_n$	the $n \times n$ identity matrix
$E_{ij}$	the elementary matrix with a 1 in the $ij$ th position, and 0 elsewhere



# Chapter 1

## Preliminaries

### 1.1 Verbal subgroups

**Definition 1.1.1.** A *word* in  $n$  variables is an expression of the form

$$x_{i_1}^{\epsilon_1} x_{i_2}^{\epsilon_2} \cdots x_{i_s}^{\epsilon_s},$$

where  $s \in \mathbb{N}_0$ , and, for each  $j = 1, \dots, s$  we have  $i_j \in \{1, 2, \dots, n\}$  and  $\epsilon_j = \pm 1$ . In the special case where  $s = 0$ , we will refer to the resulting word as the *empty word*.

Let  $w(x_1, \dots, x_n)$  be a word in  $n$  variables. Given a group  $G$  and elements  $g_1, \dots, g_n \in G$ , we will define  $w(g_1, \dots, g_n)$  to be the element of  $G$  obtained from  $w$  by replacing each instance of  $x_i$  with  $g_i$  respectively. It will be convenient to refer to such elements as either *w-values* of  $G$ , or *G-values* of  $w$ .

If  $G^{(n)}$  denotes the Cartesian product of  $G$  with itself  $n$  times, then we

can view  $w$  as a function  $G^{(n)} \rightarrow G$ . We really want to think of words in this way: we will say that words  $v$  and  $w$  in  $n$  variables are equivalent if

$$w(g_1, \dots, g_n) = v(g_1, \dots, g_n)$$

for all elements  $g_1, \dots, g_n$  of every group  $G$ . It is easy to see that  $v$  and  $w$  are equivalent if and only if  $v$  can be obtained from  $w$  by making a finite number of insertions or deletions of  $xx^{-1}$  or  $x^{-1}x$ , where  $x$  is any variable. The corresponding set of equivalence classes forms a group under concatenation; consequently we will refer to equivalent words as actually being equal. We will adopt a notation of Hanna Neumann [23], and denote the group of all words in  $n$  variables by  $X_n$ ; we will denote the group of all words (i.e. in any number of variables) by  $X_\infty$ .

**Definition 1.1.2.** Let  $w(x_1, \dots, x_n)$  be a word and  $G$  be a group. The *vocabulary* of  $w$  in  $G$  is the set of all  $w$ -values of  $G$ . That is

$$w\{G\} := \{w(g_1, \dots, g_n) \mid g_1, \dots, g_n \in G\}.$$

The *verbal subgroup* of  $w$  in  $G$  is the subgroup

$$w(G) := \langle w\{G\} \rangle$$

generated by the vocabulary of  $w$ .

Clearly  $G$  itself is a verbal subgroup, as is  $\{1\}$ . A common verbal subgroup is that of the word  $x^k$ , for some  $k \in \mathbb{N}$ . In keeping with modern texts we will

denote this verbal subgroup by  $G^k$ ; so that  $G^k$  is the subgroup of  $G$  generated by the  $k$ th powers of the elements of  $G$ .

The following elementary result is immediate from the above definitions:

**Lemma 1.1.3.** *Let  $G$  and  $H$  be groups and let  $w(x_1, \dots, x_n)$  be a word. If  $\theta : G \rightarrow H$  is a homomorphism then for any  $g_1, \dots, g_n \in G$ :*

$$w(g_1, \dots, g_n)^\theta = w(g_1^\theta, \dots, g_n^\theta).$$

Therefore  $w(G)^\theta = w(G^\theta)$ . In particular,  $w(G)$  is a fully invariant subgroup of  $G$ , and if  $K \triangleleft G$  then

$$w(G/K) = w(G)K/K.$$

**Definition 1.1.4.** Let  $G$  be a group and  $w(x_1, \dots, x_n)$  a word. An element  $m \in G$  is said to be *marginal* for  $w$  in  $G$  if

$$w(g_1, \dots, g_{i-1}, g_i m, g_{i+1}, \dots, g_n) = w(g_1, \dots, g_i, \dots, g_n)$$

for every  $g_1, \dots, g_n \in G$  and each  $i = 1, 2, \dots, n$ . The set of all elements of  $G$  which are marginal for  $w$  forms a subgroup of  $G$  called the *marginal subgroup* of  $w$ , denoted by  $w^*(G)$ .

Philip Hall posed some questions regarding the relationship between a verbal subgroup and its associated vocabulary and marginal subgroup. We will summarise a version of these questions in the definition which follows. For a complete description of the questions, see Robinson [28].

**Definition 1.1.5.** Let  $w$  be a word and let  $\mathfrak{X}$  be a class of groups. Then:

- (i)  $w$  is said to be *concise* in  $\mathfrak{X}$  if  $w(G)$  is finite whenever  $w\{G\}$  is finite, for all groups  $G \in \mathfrak{X}$ ;
- (ii)  $w$  is said to be *robust* in  $\mathfrak{X}$  if  $w(G)$  is finite whenever  $|G : w^*(G)|$  is finite, for all groups  $G \in \mathfrak{X}$ .

If  $w$  is concise in every group  $G$  then we will simply say that  $w$  is *concise*. If every word is concise in a particular group  $G$  then we will say that  $G$  is *verbally-concise*. We will use the terms *robust* and *verbally-robust* similarly.

It was originally conjectured that all words would be concise and robust; however S. Ivanov [13] has provided a counterexample describing a word which takes only one non-trivial value in some group, but which has an infinite verbal subgroup (for details of a similar counterexample see Ol'shanskii [26], Theorem 39.7).

If  $\mathcal{P}$  and  $\mathcal{Q}$  are properties of groups then we will say that a group  $G$  is  $\mathcal{P}$ -by- $\mathcal{Q}$  if  $G$  contains a normal  $\mathcal{P}$ -subgroup  $N$  such that  $G/N$  has the property  $\mathcal{Q}$ . In the special case when  $G$  is  $\mathcal{P}$ -by-finite we will say that  $G$  is *virtually*  $\mathcal{P}$ , although on occasion we will keep the term  $\mathcal{P}$ -by-finite when it aids clarity.

Philip Hall [8] has shown that all finitely generated virtually nilpotent groups are verbally-concise, and Merzljakov [22] has shown that every linear group is verbally-concise.

In general we will often find that  $w\{G\}$  is an infinite set. In this case a natural analogue to 'concise' is the concept of width.

**Definition 1.1.6.** Let  $G$  be a group generated by a set  $X$  and let  $g \in G$ . The *width* of  $g$  with respect to  $X$  is the smallest number  $l$  such that  $g$  can be written as a product of  $l$  elements of  $X$  and their inverses (that is, a group word of length  $l$  in  $X$ ). We will denote this number by  $\|g\|_X$ . If  $H$  is any subset of  $G$  then the width of  $H$  with respect to  $X$  is

$$\|H\|_X := \sup_{h \in H} \|h\|_X.$$

In practice we will drop the subscript  $X$  whenever the generating set is clear from the context.

We will denote the set of all products of  $l$  elements of  $X$  and their inverses by  $X^{*l}$ . Note that

$$\langle X \rangle = \bigcup_{i=1}^{\infty} X^{*i} = X^{*\infty}.$$

It is clear that the width of an element  $g$  with respect to any generating set is non-negative, and that  $\|g\| = 0$  exactly when  $g = 1$ . Additionally, for  $g, h \in G$ , we can see that

$$\|gh\| \leq \|g\| + \|h\|.$$

**Lemma 1.1.7.** *Let  $N$  be a normal subgroup of a group  $G$ . Let  $H = \langle X \rangle$  be a subgroup of  $G$ . Then*

$$\|H\|_X \leq \|HN/N\|_{XN/N} + \|H \cap N\|_X,$$

where  $HN/N$  and  $XN/N$  denote the natural images in  $G/N$  of  $H$  and  $X$

respectively.

*Proof.* Suppose that  $\|HN/N\|_{XN/N} = a$  and  $\|H \cap N\|_X = b$  are finite, or there is nothing to prove. Take any  $h \in H$ ; then

$$hN = (x_1N) \cdots (x_aN) = (x_1 \cdots x_a)N,$$

for some  $x_1, \dots, x_a \in X \cup X^{-1}$ . Hence

$$h = x_1 \cdots x_a n$$

for some  $n \in H \cap N$ . Therefore  $\|h\|_X \leq a + \|n\|_X$  and the result follows.  $\square$

Now let  $w$  be a word. In this work, we will always be measuring the width of  $w(G)$  with respect to  $w\{G\}$ ; thus we need only write  $\|w(G)\|$ . Furthermore, it will be convenient to refer to this value as the *width of  $w$  in  $G$* . When the group in question is clear from the context we will simply write  $\|w\|$  for the width of  $w$ .

**Lemma 1.1.8.** *Every word has width 1 in an Abelian group.*

*Proof.* Let  $w(x_1, \dots, x_n)$  be any word and let  $A$  be an Abelian group. If  $a_1, \dots, a_n, b_1, \dots, b_n \in A$  then it is easy to see that

$$w(a_1, \dots, a_n)w(b_1, \dots, b_n) = w(a_1b_1, \dots, a_nb_n);$$

whence  $w(a_1, \dots, a_n)^{-1} = w(a_1^{-1}, \dots, a_n^{-1})$ . Hence  $w\{A\}$  is a group, and  $\|w\| = 1$ .  $\square$

As a consequence of Lemma 1.1.3 we have:

**Lemma 1.1.9.** *Suppose that  $\theta : G \rightarrow H$  is a surjective homomorphism. For any word  $w$  we have*

$$\|w(H)\| \leq \|w(G)\|.$$

Additionally, Lemma 1.1.7 gives us the following helpful result:

**Lemma 1.1.10.** *If  $w$  is a word and  $N$  is a normal subgroup of a group  $G$  then*

$$\|w(G)\| \leq \|w(G/N)\| + \|w(G) \cap N\|_{w\{G\}}.$$

For most of what follows we will consider groups, or classes of groups, in which all words, or a class of words, have finite width. In keeping with the terminology in Definition 1.1.5, and adopting a term used by some previous authors (see, for example Jeremy Wilson [39]), we will use the following:

**Definition 1.1.11.** Let  $w$  be a word, and let  $\mathfrak{X}$  be a class of groups. Then  $w$  is *elliptic* in  $\mathfrak{X}$  if  $w$  has finite width in every  $G \in \mathfrak{X}$

In addition, we will say that a group  $G$  is *verbally-elliptic* if every word has finite width in  $G$ . For example, we have already seen that Abelian groups are verbally-elliptic.

Note that for a given word  $w$ , and a given class of groups  $\mathfrak{X}$ :

$$w \text{ elliptic in } \mathfrak{X} \implies w \text{ concise in } \mathfrak{X} \implies w \text{ robust in } \mathfrak{X}.$$

There will be occasions when we will need to consider verbal subgroups which are defined by a collection of words. We will end this section with a

brief consideration of this topic.

**Definition 1.1.12.** Let  $W$  be an arbitrary set of words, and let  $G$  be a group. The vocabulary of  $W$  in a group  $G$  is the set of all of the  $G$ -values of all the elements of  $W$ ; that is

$$W\{G\} := \bigcup_{w \in W} w\{G\}.$$

The verbal subgroup of  $W$  in  $G$  is the subgroup of  $G$  generated by this set.

It turns out that the study of finite collections of words can be reduced to the study of the verbal subgroups of single words. This is via a construction which I am calling the independent product:

**Definition 1.1.13.** Given two words  $v(x_1, \dots, x_m)$  and  $w(x_1, \dots, x_n)$ . We will define the *independent product* of  $v$  and  $w$  to be the word in  $m + n$  variables:

$$v \diamond w(x_1, \dots, x_m, y_1, \dots, y_n) := v(x_1, \dots, x_m)w(y_1, \dots, y_n).$$

The independent product of any number of words can then be defined inductively.

If  $W$  is a set which consists of two words  $v$  and  $w$  then it is easy to see that

$$W\{G\} \subseteq (v \diamond w)\{G\} \subseteq v(G)w(G),$$

for any group  $G$ , and so

$$W(G) \leq (v \diamond w)(G) \leq v(G)w(G).$$

But both  $v(G)$  and  $w(G)$  are contained in  $W(G)$ ; hence

$$W(G) = (v \diamond w)(G) = v(G)w(G).$$

This also shows that

$$\|v \diamond w\| \leq \|v\| + \|w\|.$$

A simple induction now provides the following:

**Lemma 1.1.14.** *Let  $W = \{w_1, \dots, w_k\}$  be a finite set of words. Then there exists a single word  $w = w_1 \diamond w_2 \diamond \dots \diamond w_k$  such that  $W(G) = w(G)$ , for all groups  $G$ . Furthermore, in any group  $\|w\| \leq \|w_1\| + \dots + \|w_k\|$ .*

In groups in which every subgroup is finitely generated this lemma also generalises to infinite collections of words; for then the values of only finitely many words will be required to generate the verbal subgroup.

## 1.2 Profinite groups

**Definition 1.2.1.** Let  $\mathcal{P}$  be a property of groups. A group  $G$  is said to be *pro- $\mathcal{P}$*  if  $G$  is the inverse limit of a surjective inverse system of groups each with  $\mathcal{P}$ .

If finite groups are bestowed with the discrete topology then a pro-finite group (typically referred to as a profinite group) is a compact Hausdorff

topological group in which the open subgroups form a base for the neighbourhoods of 1. An important subclass of the profinite groups is the pro- $p$  groups (here pro- $p$  is short for pro-(finite  $p$ -group), where  $p$  denotes some arbitrary prime).

In a finitely generated profinite group the topology is entirely determined by the group structure, with the open subgroups exactly the subgroups of finite index. In the pro- $p$  case this result is a well-known result of Serre; the more general case is a recent result of Nikolov and Segal [24]. To denote that a subgroup  $H$  of a profinite group  $G$  is open we will use the notation  $H \leq_o G$ , or  $H \triangleleft_o G$  if  $H$  is also normal. Similarly we will write  $H \leq_c G$  or  $H \triangleleft_c G$  to say that  $H$  is a closed subgroup or a closed normal subgroup of  $G$  respectively. We will denote the closure of a subset  $X \subseteq G$  by  $\overline{X}$ .

It would be very difficult to introduce all of the properties of profinite/pro- $p$  groups which we will use here, and there are textbooks devoted to this subject; this author recommends either Dixon et al [2] or Wilson [41]. Familiarity with the first four chapters of the former will be assumed from here on, although we will give the key definitions where appropriate.

When we say that a profinite group  $G$  is generated by a subset  $X$ , we mean that  $G$  is generated *topologically* by  $X$ . That is

$$G = \overline{\langle X \rangle}.$$

When we wish to discuss a subgroup which is generated *abstractly* (i.e. in the usual sense) by some set, we will specify this explicitly. A particularly

important example of this is a verbal subgroup of a profinite group.

**Definition 1.2.2.** If  $G$  is a profinite group, and  $w$  is a word, then the verbal subgroup of  $w$  in  $G$  is the subgroup of  $G$  generated abstractly by the vocabulary of  $w$  in  $G$ . That is

$$w(G) = \langle w\{G\} \rangle.$$

The question as to whether a particular verbal subgroup of a profinite group is closed is an interesting one, and one that is strongly related to the idea of width, as the following lemma shows.

**Lemma 1.2.3.** *Let  $G$  be a profinite group and let  $w$  be a word. Then the width of  $w$  in  $G$  is finite if and only if  $w(G)$  is closed in  $G$ .*

*Proof.* The vocabulary  $w\{G\}$  is the image of the Cartesian product  $G^{(n)}$  under the continuous function  $w$ ; hence  $w\{G\}$  is a closed set. But

$$w(G) = w\{G\}^{*\|w\|},$$

so if  $\|w\|$  is finite then  $w(G)$  is closed.

Conversely suppose that  $w(G)$  is closed. We have that

$$w(G) = \bigcup_{i=1}^{\infty} w\{G\}^{*i},$$

and each  $w\{G\}^{*i}$  is a closed set. Therefore, by the Baire Category Theorem, there exists  $m \geq 1$  such that  $w\{G\}^{*m}$  contains a non-empty open subset  $U$

of  $w(G)$ . Now

$$G = \bigcup_{g \in w(G)} Ug,$$

and each of the cosets  $Ug$  is open; therefore, by compactness

$$w(G) = \bigcup_{i=1}^s Ug_i,$$

for some  $g_1, \dots, g_s \in w(G)$ . If  $k$  is such that  $g_1, \dots, g_s \in w\{G\}^{*k}$ , then

$$w(G) = w\{G\}^{*(m+k)},$$

and  $\|w\| \leq m + k$ . □

Lemma 1.2.3 is one of the most important results concerning the verbal subgroups of profinite groups. For example, it helps provide us with our first example of an infinite non-Abelian verbally-elliptic profinite group: Given a finite field  $\mathbb{F}_q$ , the Nottingham Group  $\mathcal{N} := \mathcal{N}(\mathbb{F}_q)$  is the group of normalised automorphisms of the local field  $\mathbb{F}_q((t))$ . Every normal subgroup of  $\mathcal{N}$  is closed (B. Klopsch, [17]) so it follows that the Nottingham Group is verbally-elliptic. It is known that not every finitely generated pro- $p$  group is verbally-elliptic; some counterexamples are considered in the next chapter.

The example above is slightly misleading, for it is generally more useful to know that a verbal subgroup is closed than it is to know that it has finite width. The real power of Lemma 1.2.3 is that it allows us to formulate a relationship between a word's width in a profinite group  $G$ , and its width in the finite images of  $G$ :

**Proposition 1.2.4.** *Let  $w$  be a word and let  $G$  be a profinite group. Then*

$$\|w(G)\| = \sup_{N \triangleleft_o G} \|w(G/N)\|.$$

Moreover, if  $G$  is the inverse limit of an inverse system  $(G_i, \varphi_i)_{i \in I}$  of finite groups in which the maps  $\varphi_i$  are epimorphisms, then

$$\|w(G)\| = \sup_{i \in I} \|w(G_i)\|.$$

*Proof.* Observe that the width of  $w(G/N)$  for each  $N \triangleleft_o G$  is bounded by the width of  $w(G)$  by Lemma 1.1.3, so

$$\|w(G)\| \geq \sup_{N \triangleleft_o G} \|w(G/N)\|.$$

Now suppose that

$$\sup_{N \triangleleft_o G} \|w(G/N)\| = l$$

for some finite  $l$ . Let  $F = w\{G\}^{*l}$ ; so  $F$  is closed. Take any  $N \triangleleft_o G$ . We have  $w(G/N) = FN/N$  and so  $w(G)N = FN$ . Whence

$$w(G) \leq \bigcap \{FN \mid N \triangleleft_o G\} = \overline{F} = F.$$

Clearly  $F \leq w(G)$ , so  $w(G) = F$  and  $\|w(G)\| \leq l$ . The first result follows.

To see the second result, observe again that the width of each  $w(G^{\varphi_i})$  is bounded by the width of  $w(G)$ , so

$$\|w(G)\| \geq \sup_{i \in I} \|w(G_i)\|.$$

If  $N \triangleleft_o G$  then for some  $i \in I$  there exists a subset  $U \subseteq G_i$ , containing 1, such that  $\varphi_i^{-1}(U) \subseteq N$  (see Proposition 1.2.1 of [41]). Hence  $\ker \varphi_i \triangleleft_o N$ . Denoting the image of  $N$  in  $G_i$  by  $N_i$ :

$$w(G/N) \cong \frac{w(G)N/\ker \varphi_i}{N/\ker \varphi_i} \cong w(G_i)N_i/N_i = w(G_i).$$

Therefore the width of  $w(G/N)$  is bounded by that of  $w(G_i)$ . The result now follows from the first part.  $\square$

### 1.3 Profinite groups of finite rank

If  $G$  is a profinite group then denote by  $d(G)$  the size of a minimal (topological) generating set for  $G$ . If  $G$  is finite then we define the *rank* of  $G$  to be

$$\text{rk}(G) := \sup\{d(H) \mid H \leq G\}.$$

In a profinite group  $G$  the following values are equal:<sup>1</sup>

- (i)  $\sup\{d(H) \mid H \leq_c G\}$ ;
- (ii)  $\sup\{d(H) \mid H \leq_c G \text{ and } d(H) < \infty\}$ ;
- (iii)  $\sup\{d(H) \mid H \leq_o G\}$ ;
- (iv)  $\sup\{\text{rk}(G/N) \mid N \triangleleft_o G\}$ .

**Definition 1.3.1.** The *rank* of a profinite group  $G$  is the common value of the formulae above.

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<sup>1</sup>The details behind this fact are non-trivial; for details see Dixon et al [2].

If  $N$  is a closed subgroup of a profinite group  $G$  then

$$\mathrm{rk}(G) \leq \mathrm{rk}(N) + \mathrm{rk}(G/N);$$

hence a profinite group which is virtually a profinite group of finite rank is itself of finite rank.

As an illustration of how the concept of finite rank can improve our earlier results, consider the following:

**Lemma 1.3.2.** *Suppose that  $G$  is a pro- $p$  group of finite rank  $d$ , and let  $W$  be a collection of words. If every word in  $W$  has finite width in  $G$ , then the verbal subgroup  $W(G)$  has finite width. Furthermore, if the width of each word in  $W$  is bounded by  $k$ , then*

$$\|W(G)\| \leq kd.$$

*Proof.* Let  $a_1$  be any element of  $W\{G\}$ . If  $\overline{\langle a_1 \rangle}$  is not equal to  $\overline{W(G)}$  then  $\overline{\langle a_1 \rangle}$  certainly excludes an element  $a_2 \in W\{G\}$ . In this way we generate a chain of closed subgroups

$$\overline{\langle a_1 \rangle} \leq \overline{\langle a_1, a_2 \rangle} \leq \overline{\langle a_1, a_2, a_3 \rangle} \leq \cdots$$

in which each  $a_1, a_2, a_3, \dots$  is an element of  $W\{G\}$ . A pro- $p$  group of finite rank has the ascending chain condition on closed subgroups, so this process must stop; when it does we will have elements  $a_1, \dots, a_s \in W\{G\}$  which generate  $\overline{W(G)}$ .

In a  $d$ -generator pro- $p$  group every generating set contains a subset of

size at most  $d$  which is also a generating set; therefore due to the method of construction used above, it must be that  $s = d$ , and

$$\overline{W(G)} = \overline{\langle a_1, \dots, a_d \rangle}.$$

Now let  $w_1, \dots, w_d \in W$  be such that  $a_i$  is a  $w_i$ -value for each  $i = 1, \dots, d$ .

If  $w := w_1 \diamond w_2 \diamond \dots \diamond w_d$  then

$$\langle a_1, \dots, a_d \rangle \leq w(G) \leq W(G).$$

If  $l_i$  denotes the width of  $a_i$  with respect to  $W\{G\}$  then, by Lemma 1.1.14, the width of  $w(G)$  with respect to  $W\{G\}$  is no greater than  $l_1 + \dots + l_d$ ; so  $w(G)$  is closed, by Lemma 1.2.3. Consequently  $w(G) = W(G)$ , and

$$\|W(G)\| \leq l_1 + \dots + l_d.$$

Clearly, if the width of every word in  $W$  is bounded by  $k$ , then  $\|W(G)\| \leq kd$ . □

A profinite group of finite rank does not necessarily have the ascending chain condition on closed subgroups, so the above argument cannot be used for collections of words in this case. If, however,  $G$  is a profinite group which is virtually a pro- $p$  group of finite rank then  $G$  *does* have the ascending chain condition on closed subgroups; therefore parts of the above proof can be easily be restated to give the following result:

**Lemma 1.3.3.** *Suppose that  $G$  is a profinite group which is virtually a pro- $p$  group of finite rank, and let  $W$  be a collection of words. If every word in  $W$  has finite width in  $G$ , then the verbal subgroup  $W(G)$  has finite width.*

Unfortunately the proof of Lemma 1.3.2 does not generalise sufficiently to provide us with a bound for the width of a verbal subgroup in this generalised case.

Pro- $p$  groups of finite rank are linear over  $\mathbb{Z}_p$  (see Theorem 7.19 of Dixon et al [2]), and any virtually linear group is also linear (by taking the induced representation). Since every linear group is verbally-concise (by Merzljakov [22]) we have the following interesting result:

**Lemma 1.3.4.** *Every profinite group which is virtually a pro- $p$  group of finite rank is verbally-concise.*

**Remark.** A profinite group which is virtually a pro- $p$  group of finite rank is often referred to as a *compact  $p$ -adic analytic* group. The main result of this work appears in Chapter 4, where we extend Lemma 1.3.4 to show that such groups are verbally-elliptic. To prove this result we will only use the fact that these groups contain an open normal pro- $p$  group of finite rank. Hence we will continue to refer to such profinite groups as virtually pro- $p$  of finite rank.

In Chapter 2 we will prove that the so-called *outer-commutator* words have finite width in all profinite groups of finite rank. This class of groups is larger than the one discussed in the previous paragraph; however the only property which we will exploit is that every closed subgroup is finitely generated. Notice from Definition 1.3.1 that it is not clear whether this latter

property is sufficient to imply that a group has finite rank. Indeed this is an open problem, but one that is expected to have a positive solution. With this in mind, we will always state that a profinite group should have finite rank, even when the only property we need is that every closed subgroup is finitely generated.

## 1.4 Notes

Most of the material in the chapter is ‘folklore’. Supplementary material regarding verbal subgroups and marginal subgroups can be found in the original papers of Philip Hall [6], P. W. Stroud [34] and R. F. Turner-Smith [36, 35]. A detailed survey can also be found in Robinson [28].

In [36] Turner-Smith shows that in a residually-finite group, a word is concise if and only if it is robust. He also shows that a group  $G$  is verbally-concise if every homomorphic image of  $G$  is residually finite. It is an open problem whether every residually finite group is verbally-concise. Because the proofs and counterexamples in this field of study are complex, it is hard to get a feel for whether this problem will have a positive or negative solution.

# Chapter 2

## Commutator words

### 2.1 Introduction

One of the simplest, most useful, and most common words is the commutator

$$[x_1, x_2] := x_1^{-1}x_2^{-1}x_1x_2.$$

The verbal subgroup of the commutator is the familiar *derived group*  $[G, G]$ , which is often denoted as  $G'$ . More generally, if  $H$  and  $K$  are subgroups of some group then the commutator of  $H$  and  $K$  is the group

$$[H, K] := \langle [h, k] \mid h \in H, k \in K \rangle.$$

However, if  $h \in H$  then by  $[h, K]$  we will mean the *set*

$$[h, K] := \{[h, k] \mid k \in K\}.$$

As usual we will write commutators with the bracketing associated to the left, so that

$$[x_1, \dots, x_n] := [[x_1, \dots, x_{n-1}], x_n].$$

**Definition 2.1.1.** A word  $w$  is called a *commutator word* if  $w$  is an element of the group

$$[X_\infty, X_\infty].$$

Clearly if  $w$  is a commutator word and  $G$  is a group then  $w(G) \leq G'$ . The converse of this is not true (consider any group in which  $G' = G$ ), but if  $w(G) \leq G'$  for *every* group  $G$  then  $w$  must be a commutator word.

There are three important identities which we will regularly use when working with the commutator:

$$ba = ab[b, a], \tag{2.1}$$

$$[ab, c] = [a, c]^b [b, c], \tag{2.2}$$

$$[a, bc] = [a, c][a, b]^c. \tag{2.3}$$

The first of these can be used to ‘collect’ the instances of each variable in a word  $w(x_1, \dots, x_n)$  and write  $w$  in the form

$$w = x_1^{\beta_1} \cdots x_n^{\beta_n} \lambda, \tag{2.4}$$

where  $\lambda$  is a commutator word and  $\beta_1, \dots, \beta_n$  are integers. It is clear that  $w$  is a commutator word if and only if  $\beta_1 = \dots = \beta_n = 0$ .

**Definition 2.1.2.** Let  $v(x_1, \dots, x_m)$  and  $w(x_1, \dots, x_n)$  be words. The *independent commutator* of  $v$  and  $w$  is the word  $[v | w]$ , defined by

$$[v | w](x_1, \dots, x_m, y_1, \dots, y_n) := [v(x_1, \dots, x_m), w(y_1, \dots, y_n)].$$

We can use Equations (2.2) and (2.3) to see that a typical element of the group  $[v(G), w(G)]$  can be written as a product of elements of the form  $[x, y]^g$ , where  $x$  is a  $v$ -value,  $y$  is a  $w$ -value and  $g \in G$ . By Lemma 1.1.3 each of these elements is in  $[v | w]\{G\}$ . It is also clear that  $[v | w](G) \leq [v(G), w(G)]$ . Hence

$$[v | w](G) = [v(G), w(G)].$$

We may use the independent commutator to construct an important family of commutator words:

**Definition 2.1.3.** We will define the *outer-commutator* words inductively as follows: The outer-commutator word of degree 0 is  $x_1$ . If  $v$  and  $w$  are outer-commutator words of degrees  $m$  and  $n$  respectively, then  $[v | w]$  is an outer-commutator word of degree  $m + n + 1$ .

The definition of degree has been chosen to reflect the total number of ‘commutations’ which need to be carried out to obtain the word. The number of variables occurring in an outer-commutator word is always one more than the degree of the word. Note that the outer-commutator word of degree 0 is not actually a commutator word.

Special cases of outer-commutator words are the familiar *lower central*

words

$$\gamma_1(x_1) := x_1, \quad \gamma_n(x_1, \dots, x_n) := [\gamma_{n-1} \mid \gamma_1],$$

and the *derived* words

$$\delta_1(x_1, x_2) := [x_1, x_2], \quad \delta_n(x_1, \dots, x_{2^n}) := [\delta_{n-1} \mid \delta_{n-1}],$$

the verbal subgroups of which are the lower central series, and the derived series, respectively.

The outer-commutator words are concise (Turner-Smith, [36]), but not elliptic. During the preparation of this document, Andrei Jaikin-Zapirain announced the following ‘mega-theorem’:

**Theorem 2.1.4** (Jaikin-Zapirain, [14]). *A word has finite width in every finitely generated pro- $p$  group if and only if it is not an element of*

$$X''_{\infty}(X'_{\infty})^p.$$

Although this confirms that the lower central words have finite width in finitely generated pro- $p$  groups, Nikolov and Segal [25] have in fact shown that these words have finite width in all finitely generated *profinite* groups.

Many of the other outer-commutator words are contained in the above group, so they do not have finite width in all finitely generated pro- $p$  groups. However, Nikolov and Segal have also shown that the derived words have finite width in all profinite groups of finite rank. The main result of this chapter is to extend this, and prove that every outer-commutator word has

finite width in a profinite group of finite rank. We will also provide some bounds for this width.

Theorem 2.1.4 clearly shows that every word which is not a commutator word has finite width in every finitely generated pro- $p$  group. We make the following conjecture:

**Conjecture 1.** *Every word which is not a commutator word has finite width in every finitely generated profinite group.*

In fact, Conjecture 1 is equivalent to a result about a much simpler class of words, as follows:

**Lemma 2.1.5.** *Let  $G$  be a finitely generated profinite group. Then all of the words which are not commutator words have finite width in  $G$  if and only if the ‘Burnside’ words  $x^k$ ,  $k = 1, 2, \dots$ , have finite width in  $G$ .*

*Proof.* If every word which is not a commutator word has finite width in every finitely generated profinite group, then certainly so too will each of the Burnside words.

Conversely, let  $G$  be a  $d$ -generator profinite group, and suppose that each of the Burnside words has finite width in  $G$ . Let  $w(x_1, \dots, x_n)$  be a word which is not a commutator word. From Equation (2.4), we can write

$$w = x_1^{\beta_1} \cdots x_n^{\beta_n} \lambda,$$

where at least one of  $\beta_1, \dots, \beta_n$  is non-zero. Let’s say that one such non-zero value is  $k$ ; then  $w(G)$  contains  $G^k$ . Now  $G^k$  has finite width by hypothesis; therefore  $G^k$  is closed by Lemma 1.2.3.

The profinite group  $G/G^k$  is the inverse limit of a system of  $d$ -generator finite groups of exponent no greater than  $k$ , and by the Restricted Burnside Problem there are only finitely many such groups; therefore  $G/G^k$  is finite. It follows that  $G^k$  is in fact open, and consequently  $w(G)$  is open (and closed). Hence  $w(G)$  has finite width by Lemma 1.2.3.  $\square$

C. Martinez [21] has shown that the Burnside words have finite width in all finitely generated pronilpotent groups, and Nikolov and Segal have shown that the Burnside words have finite width in all finitely generated profinite groups which do not involve every finite group as an open section. The general status of Conjecture 1 is still open, but I expect that it will have a positive solution.

## 2.2 A two-variable counterexample

The second derived word  $\delta_2 := [[x_1, x_2], [x_3, x_4]]$  does not have finite width in every finitely generated pro- $p$  group. Although this result falls within the scope of Theorem 2.1.4, it has been known since 1982, when Roman'kov [29] provided an example of a finitely generated soluble pro- $p$  group  $\mathfrak{R}$  in which  $\delta_2$  has infinite width. In this section we will describe the construction of  $\mathfrak{R}$ , with a view to proving that the word  $[[x, y], [x, y, y]]$  also has infinite width in this group.

Fix a prime number  $p$  and write  $\mathbb{F}_p$  for the field with  $p$  elements. Let  $C := \langle a, b \rangle$  be the 2-generator free Abelian group and let  $A := \langle a \rangle$  and  $B := \langle b \rangle$ . For each  $k \in \mathbb{N}$  let  $A_k := A/A^{p^k}$ ,  $B_k := B/B^{p^k}$ , and  $C_k := C/C^{p^k}$ ; for simplicity we will also denote the natural generators of  $A_k$  and  $B_k$  by

$a$  and  $b$  respectively. We will refer to the elements of the respective group algebras  $\mathbb{F}_p A_k$ ,  $\mathbb{F}_p B_k$ , and  $\mathbb{F}_p C_k$  as polynomials.

**Definition 2.2.1.** Any polynomial  $h(a, b) \in \mathbb{F}_p C_k$  can be written in the form

$$h(a, b) = \sum_{i=1}^n f_i(a)g_i(b),$$

where the  $f_i \in \mathbb{F}_p A_k$  are polynomials in  $a$  only, and the  $g_i \in \mathbb{F}_p B_k$  are polynomials in  $b$  only. The *splitting degree* of  $h$  is the smallest number  $n$  such that  $h$  can be written in the above form.

In general, determining bounds for the width of a verbal subgroup, especially lower bounds, is difficult. Roman'kov's idea was to simplify this by reducing the study of width to the study of splitting degree. We can do this in the finite group  $\mathfrak{R}_k$  consisting of elements of the form

$$\begin{pmatrix} a^i & & \\ f(a) & 1 & \\ h(a, b) & g(b) & b^j \end{pmatrix},$$

where  $f \in \mathbb{F}_p A_k$ ,  $g \in \mathbb{F}_p B_k$ ,  $h \in \mathbb{F}_p C_k$ , and  $i, j \in \mathbb{Z}/p^k\mathbb{Z}$ . We can calculate

$$|\mathfrak{R}_k| = p^{2k+2p^k+p^{2k}},$$

and it can be shown that  $\mathfrak{R}_k$  is generated by the elements  $E_{21}$ ,  $E_{32}$  and the diagonal matrix  $D = \begin{pmatrix} a & & \\ 0 & 1 & \\ 0 & 0 & b \end{pmatrix}$ .

The main observation for us to make is that the elements of the derived group of  $\mathfrak{R}_k$  have the form

$$\begin{pmatrix} 1 & & \\ f(a) & 1 & \\ h(a, b) & g(b) & 1 \end{pmatrix},$$

where  $f$ ,  $g$  and  $h$  are some elements of  $\mathbb{F}_p A_k$ ,  $\mathbb{F}_p B_k$  and  $\mathbb{F}_p C_k$  respectively.<sup>1</sup>

The commutator of any two of these elements is

$$\left[ \begin{pmatrix} 1 & & \\ f_1 & 1 & \\ h_1 & g_1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & & \\ f_2 & 1 & \\ h_2 & g_2 & 1 \end{pmatrix} \right] = \begin{pmatrix} 1 & & \\ 0 & 1 & \\ f_2 g_1 - f_1 g_2 & 0 & 1 \end{pmatrix}, \quad (2.5)$$

and the splitting degree of the 3,1 entry here is at most 2. If  $w_1$  and  $w_2$  are commutator words, and  $w = [w_1, w_2]$ , then the  $\mathfrak{R}_k$ -values of  $w$  will be of the form of Equation (2.5), and  $w(\mathfrak{R}_k)$  will consist of elements of the form

$$I_3 + h(a, b)E_{31}; \quad (2.6)$$

therefore the width of this element will be at least half the splitting degree of  $h(a, b)$ .

**Lemma 2.2.2.** *For each  $k$  the polynomial*

$$(a-1)^2(b-1)(ab + a^2b^2 + \cdots + a^{p^k-3}b^{p^k-3}),$$

*viewed as an element of  $\mathbb{F}_p C_k$ , has splitting degree  $p^k - 3$ .*

---

<sup>1</sup>It is not possible for  $f$ ,  $g$  and  $h$  to be *any* polynomials, but the form to which they are restricted will turn out to be irrelevant for our discussion.

*Proof.* Write  $h$  for the above polynomial and suppose that  $h$  has splitting degree  $m$ . Clearly the splitting degree of  $h$  is no greater than  $p^k - 3$ , so  $m \leq p^k - 3$ .

Because the splitting degree of  $h$  is equal to  $m$ , there exist polynomials  $f_1, \dots, f_m \in \mathbb{F}_p A_k$  and polynomials  $g_1, \dots, g_m \in \mathbb{F}_p B_k$  such that

$$h(a, b) = f_1(a)g_1(b) + \dots + f_m(a)g_m(b).$$

The highest power of  $a$  or  $b$  which appears in  $h$  is  $p^k - 1$ , so for each  $i = 1, \dots, m$  and  $j = 1, \dots, p^k - 1$  let  $\alpha_{ij}, \beta_{ij} \in \mathbb{F}_p$  be such that

$$\begin{aligned} f_i(a) &= \alpha_{i1}a + \dots + \alpha_{ip^k-1}a^{p^k-1}; \\ g_i(b) &= \beta_{i1}b + \dots + \beta_{ip^k-1}b^{p^k-1}. \end{aligned}$$

Write  $F = (\alpha_{ij})$  and  $G = (\beta_{ij})$  and let  $H = F^T G$ . Therefore  $H$  is a  $(p^k - 1) \times (p^k - 1)$  matrix with rank at most  $m$ .

We will now show that the rank of  $H$  is equal to  $p^k - 3$ , so that  $m \geq p^k - 3$ , which will complete the proof: Notice that

$$(H)_{st} = \sum_{k=1}^m \alpha_{ks} \beta_{kt}.$$

This entry is equal to the coefficient in  $h$  of the term  $a^s b^t$ ; in this sense  $H$  is a matrix which represents  $h$ .

Let  $H_1$  be the matrix whose entries represent the factor

$$ab + a^2 b^2 + \dots + a^{p^k-3} b^{p^k-3}$$

which appears in  $h$ . That is  $H_1$  is a  $(p^k - 1) \times (p^k - 1)$  diagonal matrix with all of the diagonal entries equal to 1, except for the last two which are equal to 0. This matrix clearly has rank equal to  $p^k - 3$ .

Let  $H_2$  be the matrix which represents the polynomial

$$(b - 1)(ab + a^2b^2 + \dots + a^{p^k-3}b^{p^k-3}).$$

This can be obtained by multiplying  $H_1$  on the left by the  $(p^k - 1) \times (p^k - 1)$  matrix which has diagonal entries which are all equal to  $-1$ , and entries above the diagonal which are equal to 1. This matrix clearly has rank  $p^k - 1$ ; therefore  $H_2$  has rank  $p^k - 3$ .

Now let  $H_3$  be the matrix which represents the polynomial

$$(a - 1)(b - 1)(ab + a^2b^2 + \dots + a^{p^k-3}b^{p^k-3}).$$

This can be obtained by multiplying  $H_2$  on the left by the  $(p^k - 1) \times (p^k - 1)$  matrix in which the diagonal entries are equal to  $-1$ , and the entries below the diagonal are equal to 1. This matrix has rank  $p^k - 1$ ; therefore  $H_3$  has rank  $p^k - 3$ .

The matrix  $H$  is now obtained by multiplying  $H_3$  on the left by the same matrix described in the above paragraph. Therefore  $H$  has rank  $p^k - 3$ . This completes the proof.  $\square$

**Remark.** The above proof illustrates a general method which I believe is the most efficient way of proving things about splitting degree. The choice of  $p^k - 3$  was made for the power to avoid the complication that  $a^{p^k} = 1$ ,

but other than this all proofs about splitting degree can be produced in the same way: If  $H$  is a matrix which represents some polynomial  $h$ , then the above shows that the rank of  $H$  is at most equal to the splitting degree of  $h$ . This only leaves the splitting degree to be bounded above, which is usually an easy task.

Our main strategy for developing our counterexample is now clear, we will show that polynomials of the above form can appear as the polynomials  $h(a, b)$  in Equation (2.6).

**Lemma 2.2.3.** *The word  $w(x, y) := [[x, y], [x, y, y]]$  has width at least  $\frac{1}{2}(p^k - 3)$  in the group  $\mathfrak{A}_k$ .*

*Proof.* For each  $t = 1, \dots, p^k - 3$  let

$$X_t = \begin{pmatrix} 1 & & \\ a^t & 1 & \\ 0 & 0 & b \end{pmatrix}, \quad Y_t = \begin{pmatrix} a & & \\ 0 & 1 & \\ 0 & b^{t+1} & 1 \end{pmatrix},$$

and write  $Z_t = w(X_t, Y_t)$ . We may calculate<sup>2</sup> that

$$Z_t = I_3 + (a - 1)^2(b - 1)a^t b^t E_{31}.$$

The elements  $Z_t$  are in the vocabulary of  $w$ , so  $w(\mathfrak{A}_k)$  must contain the

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<sup>2</sup>For the reader wishing to verify this calculation, I suggest that you ignore the 3,1 entry of your matrices until the final calculation (c.f. Equation (2.5)) and that you resist the temptation to spend time simplifying the other entries—it will be eventually clear without simplification that most become zero.

element

$$Z_1 Z_2 \cdots Z_{p^k-1} = I_3 + (a-1)^2(b-1)(ab + a^2b^2 + \cdots + a^{p^k-3}b^{p^k-3})E_{31}.$$

Equation (2.5) shows that the elements of  $w\{\mathfrak{R}_k\}$  have a 3,1 entry with splitting degree at most 2; hence  $w(\mathfrak{R}_k)$  has width at least  $\frac{1}{2}(p^k - 3)$ .  $\square$

All that remains is for us to ‘put together’ the groups  $\mathfrak{R}_k$  to form a pro- $p$  group. For each  $k$  there is an obvious epimorphism  $\mathfrak{R}_{k+1} \rightarrow \mathfrak{R}_k$ , and the  $\mathfrak{R}_k$  form an inverse system under these maps. Let us denote the inverse limit of this system by  $\mathfrak{R}$ ; then  $\mathfrak{R}$  is a finitely generated pro- $p$  group.

Together with Proposition 1.2.4, Lemma 2.2.3 now proves:

**Theorem 2.2.4.** *The word  $w(x_1, x_2) = [[x_1, x_2], [x_1, x_2, x_2]]$  has infinite width in  $\mathfrak{R}$ .*

## 2.3 The commutator of closed subgroups

When a commutator word has finite width in a profinite group, its verbal subgroup is closed. This suggests the more general question: If  $H$  and  $K$  are closed subgroups of a profinite group  $G$ , is the commutator  $[H, K]$  also closed in  $G$ ? A well-known result, the original argument for which is generally attributed to Stroud, states:

**Proposition 2.3.1.** *Let  $G = \overline{\langle a_1, \dots, a_d \rangle}$  be a finitely generated pro-nilpotent group. Then*

$$[G, G] = [a_1, G][a_2, G] \cdots [a_d, G].$$

This will be proved below.

So  $[G, G]$  has finite width with respect to its natural generating set, and is closed. Let us denote

$$\mathfrak{C}\{H, K\} = \{[h, k] \mid h \in H, k \in K\}$$

for the natural generating set of  $[H, K]$ . When we refer to the width of  $[H, K]$  we will, by default, mean the width of  $[H, K]$  with respect to  $\mathfrak{C}\{H, K\}$ .

Nikolov and Segal [25] have proved:

**Theorem 2.3.2** (Nikolov–Segal). *There exists a function  $f(d)$  such that if  $G$  is a  $d$ -generator profinite group and  $H \triangleleft_c G$  then*

$$\|[H, G]\| \leq f(d).$$

Nikolov and Segal report that  $f(d)$  may be taken to be  $12d^3 + \mathcal{O}(d^2)$  in general, and that  $f(d)$  may be taken to be  $72d^2 + 46d$  when  $G$  is pro-soluble.

We cannot drop the requirement that  $G$  is finitely generated from Theorem 2.3.2: The derived group  $\mathfrak{R}'$  is not finitely generated, and Roman'kov's original counterexample shows that  $[\mathfrak{R}', \mathfrak{R}']$  is not closed. It is also necessary that  $H$  be normal in  $G$ , and we can again find a counterexample in  $\mathfrak{R}$ :

Let  $H_k$  be the subgroup of  $\mathfrak{R}_k$  consisting of all elements of the form

$$\begin{pmatrix} 1 & & & \\ f(a) & 1 & & \\ 0 & 0 & 1 & \end{pmatrix},$$

with  $f(a) \in \mathbb{F}_p A_k$ . In this case  $H_k$  is not normal in  $\mathfrak{R}_k$ . A typical element of  $\mathfrak{C}\{H_k, \mathfrak{R}_k\}$  has the form

$$\left[ \begin{pmatrix} 1 & & \\ f_1(a) & 1 & \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} a^i & & \\ f_2(a) & 1 & \\ h(a) & g(a) & b^j \end{pmatrix} \right] = \begin{pmatrix} 1 & & \\ (a^i - 1)f_1(a) & 1 & \\ -a^i b^{-j} f_1(a)g(b) & 0 & 1 \end{pmatrix};$$

so the 3,1 entry has splitting degree at most 1. Setting  $i, j = 0$ ,  $f_1(a) = a^t$  and  $g(a) = -b^t$  we see that  $\mathfrak{C}\{H_k, \mathfrak{R}_k\}$  contains the elements

$$Z_t = \begin{pmatrix} 1 & & \\ 0 & 1 & \\ a^t b^t & 0 & 1 \end{pmatrix}.$$

It is now clear, by the arguments of section 2.2, that  $[H_k, \mathfrak{R}_k]$  has width at least  $p^k - 1$ . If  $H$  is the inverse limit of the  $H_k$ , then  $H$  is a closed subgroup of  $\mathfrak{R}$ , and  $[H, \mathfrak{R}]$  does not have finite width; therefore  $[H, \mathfrak{R}]$  is not closed.

We have shown that we cannot drop any of the conditions of Theorem 2.3.2, but we can generalise this theorem to the commutator  $[H, K]$  of finitely generated closed normal subgroups  $H$  and  $K$ . We will prove this *using* Theorem 2.3.2, once we have developed a little extra machinery.

In the following lemma we will be using our usual notation for words to describe the elements of a group in terms of a generating set. In this case, words are being used for notational convenience only; the lemma does not concern words in the sense used throughout the rest of this document.

**Lemma 2.3.3.** *Let  $G$  be a group. Suppose that  $H \leq G$  is a finitely generated subgroup and that  $K \leq G$ . If  $[H, K]$  is central in  $G$  then*

$$[H, K] = [h_1, K] \cdots [h_d, K],$$

where  $h_1, \dots, h_d$  are semigroup generators of  $H$ .

*Proof.* A typical element of  $[H, K]$  has the form

$$[v_1(h_1, \dots, h_d), k_1] \cdots [v_s(h_1, \dots, h_d), k_s]$$

for some  $s \geq 1$ , where  $v_1, \dots, v_s$  are group words, and  $k_1, \dots, k_s$  are some elements of  $K$ . By the linearity of the commutators in their first variable, we may rewrite this as

$$v_1([h_1, k_1], \dots, [h_d, k_1]) \cdots v_s([h_1, k_s], \dots, [h_d, k_s]).$$

Because these elements commute, we can rearrange them so that those containing  $h_1$  are together, and similarly for  $h_2, \dots, h_d$ . This gives an expression of the form

$$w_1([h_1, k_1], \dots, [h_1, k_s]) \cdots w_d([h_d, k_1], \dots, [h_d, k_s]),$$

where  $w_1, \dots, w_d$  are group words. Collecting these commutators together using linearity in the second variable now shows that our element may be written as

$$[h_1, w_1(k_1, \dots, k_s)] \cdots [h_d, w_d(k_1, \dots, k_s)],$$

which is an expression of the required form.  $\square$

A simple consequence of this is:

**Lemma 2.3.4.** *Let  $G = \langle a_1, \dots, a_d \rangle$  be a nilpotent group. Then*

$$[G, G] = [a_1, G] \cdots [a_d, G].$$

*Proof.* We proceed by induction on the class  $c$  of  $G$ , the case  $c = 1$  being clear. By hypothesis

$$[G, G] = [a_1, G] \cdots [a_d, G] \gamma_c(G),$$

and by the previous lemma, with  $H = G$  and  $K = \gamma_{c-1}(G)$ , it follows that

$$\gamma_c(G) = [G, \gamma_{c-1}(G)] = [a_1, \gamma_{c-1}(G)] \cdots [a_d, \gamma_{c-1}(G)].$$

But each  $[a_i, \gamma_{c-1}(G)]$  is central in  $G$ , so the result follows.  $\square$

The reader may like to note that Proposition 2.3.1 follows immediately from this Lemma using Proposition 1.2.4.

**Theorem 2.3.5.** *Let  $G$  be a profinite group. Suppose that  $H \triangleleft_c G$  is a  $d_1$ -generator subgroup and  $K \triangleleft_c G$  is a  $d_2$ -generator subgroup, with  $d_1 \leq d_2$ . Then the width of  $[H, K]$  is bounded by*

$$f(d_1) + f(d_2) + d_1,$$

where  $f$  is as in Theorem 2.3.2. In particular  $[H, K] \triangleleft_c G$ .

*Proof.* Without loss of generality suppose that  $G = HK$ , so  $G$  is finitely generated. Let  $D := H \cap K$ ; then  $[D, G]$  is closed by Theorem 2.3.2. Our strategy will be to show that  $[H, K]$  has finite width modulo  $[D, G]$ , and that  $[D, G]$  has finite width with respect to  $\mathfrak{C}\{H, K\}$ .

Write  $G_1 := G/[D, G]$  and denote the respective images of  $H$  and  $K$  in  $G_1$  by  $H_1$  and  $K_1$ . Note that  $[H_1, K_1]$  is central in  $G_1$ . Let  $N \triangleleft_o G_1$ , so  $H_1N/N$  is a  $d_1$ -generator finite group. Lemma 2.3.3 shows that  $[H_1N/N, K_1N/N]$  has width  $\leq d_1$ . Our choice of  $N$  was arbitrary, so  $[H_1, K_1]$  has width  $\leq d_1$  by Proposition 1.2.4.

We can write

$$[D, G] = [D, H][D, K].$$

Theorem 2.3.2 tells us that  $[D, H]$  has width no greater than  $f(d_1)$  with respect to  $\mathfrak{C}\{D, H\}$ , and that  $[D, K]$  has width no greater than  $f(d_2)$  with respect to  $\mathfrak{C}\{D, K\}$ . But  $\mathfrak{C}\{D, K\} \subseteq \mathfrak{C}\{H, K\}$  and the inverses of the elements of  $\mathfrak{C}\{D, H\}$  are contained in  $\mathfrak{C}\{H, K\}$ . Therefore the width of  $[D, G]$  with respect to  $\mathfrak{C}\{H, K\}$  is bounded by  $f(d_1) + f(d_2)$ . Our result now follows by Lemma 1.1.7.  $\square$

In the next section we will consider some important consequences of this theorem. We will conclude this section by showing that the condition that  $H$  and  $K$  be normal is necessary for the above theorem to hold.

Let  $C_{p^k} = \langle a \rangle$  be the cyclic group of order  $p^k$ . Let  $\mathbb{F}_p$  denote the field with  $p$  elements and view the group algebra  $\mathbb{F}_p C_{p^k}$  as an additive Abelian group. There is an obvious action of  $C_{p^k}$  on this group, and we may form

the corresponding semidirect product

$$G_k := \mathbb{F}_p C_{p^k} \rtimes C_{p^k};$$

that is,  $G_k$  is the wreath product  $C_p \wr C_{p^k}$ . For now we will view this as an external semidirect product in order to avoid confusion between, for example, the element  $\lambda a \in \mathbb{F}_p C_{p^k}$  which corresponds in  $G_k$  to the pair  $(\lambda a, 1)$ , and the element  $\lambda a \in G_k$  corresponding to the pair  $(\lambda, a)$ . Note that  $|G_k| = p^{k+p^k}$  and that  $G_k$  is generated by the elements  $(1, 1)$  and  $(0, a)$ .

**Lemma 2.3.6.** *Let  $H_k := \langle (1, 1) \rangle$ . Then  $[H_k, G_k]$  has width at least  $p^k - 1$ .*

*Proof.* Let  $\lambda \in \mathbb{F}_p$ ,  $x \in \mathbb{F}_p C_{p^k}$  and  $a^n \in C_{p^k}$ . Then a typical element of  $\mathfrak{C}\{H_k, G_k\}$  has the form

$$[(\lambda, 1), (x, a^n)] = (\lambda(a^n - 1), 1).$$

Because these elements lie entirely in  $\mathbb{F}_p C_{p^k}$ , we will now drop the external notation. Therefore  $[H_k, G_k]$  is the subgroup of  $\mathbb{F}_p C_{p^k}$  generated by the elements  $(a^n - 1)$ , for  $a^n \in C_{p^k}$ . These generators are free generators (for we have actually found that  $[H_k, G_k]$  is the augmentation ideal of  $\mathbb{F}_p C_{p^k}$ ); so the expression

$$(a - 1) + (a^2 - 1) + \cdots + (a^{p^k-1} - 1)$$

has width  $p^k - 1$  with respect to  $\mathfrak{C}\{H_k, G_k\}$ . □

Taking the inverse limits of the  $G_k$  and  $H_k$  under the obvious epimorphisms, and combining this result with Proposition 1.2.4, now proves:

**Proposition 2.3.7.** *There is a 2-generator pro- $p$  group  $G$ , containing a closed procyclic subgroup  $H$ , such that  $[H, G]$  is not closed.*

In Theorem 2.3.2 we saw that the result of Theorem 2.3.5 still held true when  $H$  was not finitely generated, provided that  $K = G$ . The following, however, is an open problem:

**Question 1.** *Let  $G$  be a profinite group with  $H, K \triangleleft_c G$ . If  $K$  is finitely generated, then is  $[H, K]$  closed?*

I expect that this will not be true in general, and that a counterexample should exist along similar lines to the one illustrated above.

## 2.4 Outer-commutator words

In Chapter 4 we will see that every word has finite width in a pro- $p$  group of finite rank. The techniques we will use to do this do not work generally for profinite groups of finite rank. In fact, it is questionable whether we should expect this result to be true in such groups. The case of independent commutators, and in particular the outer-commutator words, is quite a bit simpler, and we will consider such words in this section.

By applying a routine induction argument to Proposition 2.3.1 (a similar argument is detailed in the proof of Corollary 2.4.1) we can show that the derived words have finite width in finite rank pro- $p$  groups. By similar means, Theorem 2.3.2 can be used to show that both the derived words and the lower central words have finite width in finite rank *profinite* groups. Theorem 2.3.5 can take us further:

**Corollary 2.4.1.** *Let  $G$  be a profinite group of finite rank. If  $w_1$  and  $w_2$  have finite width in  $G$ , then so does the independent commutator  $[w_1 | w_2]$ .*

*Proof.* Let  $w := [w_1 | w_2]$ . The verbal subgroups  $w_1(G)$  and  $w_2(G)$  are both closed; therefore they are finitely generated. This means that  $[w_1(G), w_2(G)]$  is closed. But this subgroup is equal to  $w(G)$ , and so  $\|w(G)\| < \infty$ .  $\square$

**Corollary 2.4.2.** *Every outer-commutator word has finite width in every profinite group of finite rank.*

*Proof.* We proceed by induction on the degree of the outer-commutator word. The outer-commutator word of degree 0 is just  $x_1$ , which clearly has finite width in any group.

Now suppose that  $w$  is an outer-commutator word of degree  $t > 0$  and that  $G$  is a profinite group of finite rank. By definition there exist outer-commutator words  $u$  and  $v$  of degrees  $r$  and  $s$  respectively such that  $w = [u | v]$ , and  $r + s + 1 = t$ . By hypothesis both  $u$  and  $v$  have finite width in  $G$  so by Corollary 2.4.1 so too does  $w$ .  $\square$

We can go even further than Corollary 2.4.1, and calculate an upper bound for the width of an independent commutator  $w := [w_1 | w_2]$  in terms of the width of  $w_1$  and  $w_2$ : Suppose that the rank of  $G$  is  $d$ ; so Theorem 2.3.5 shows that the width of  $w(G)$  with respect to  $\mathfrak{C}\{w_1(G), w_2(G)\}$  is bounded by  $2f(d) + d$ . Let us write this value as  $F(d)$ . We must now calculate a bound for the width of  $\mathfrak{C}\{w_1(G), w_2(G)\}$  with respect to the vocabulary of  $w$ .

A typical element of  $\mathfrak{C}\{w_1(G), w_2(G)\}$  has the form  $[a, b]$ , where  $a$  is a product of  $k$   $w_1$ -values and their inverses, and  $b$  is a product of  $l$   $w_2$ -values and their inverses, for some  $k, l \in \mathbb{N}$ . We may use the identities (2.2) and (2.3) to write  $[a, b]$  as a product of  $kl$  elements of the form

$$[u_i^{\pm 1}, v_j^{\pm 1}],$$

with each  $u_i$  a  $w_1$ -value and each  $v_j$  a  $w_2$ -value. Note that the elements  $u_i^{\pm 1}$  here will not be the original  $w_1$ -values appearing in the product for  $a$ , they will be conjugates of them. The same thing applies for the elements  $v_j^{\pm 1}$ . The elements of the form  $[u_i, v_j]$  are clearly elements of  $w\{G\}$ , and we can use the identities

$$\begin{aligned} [u^{-1}, v] &= ({}^u[u, v])^{-1} = [{}^u u, {}^u v]^{-1}, \\ [u, v^{-1}] &= ({}^b[u, v])^{-1} = [{}^b u, {}^b v]^{-1}, \\ [u^{-1}, v^{-1}] &= {}^{uv}[u, v] = [{}^{uv} u, {}^{uv} v], \end{aligned}$$

to see that all of the  $[u_i^{\pm 1}, v_j^{\pm 1}]$  are elements of  $w\{G\}$  or their inverses. Now  $k$  and  $l$  are bounded by  $\|w_1\|$  and  $\|w_2\|$ ; hence

$$\|w(G)\| \leq F(d)\|w_1\|\|w_2\|.$$

In the case of outer-commutator words, this leads to a particularly pleasing result, which can be established by a routine induction, similar to that in the proof of Corollary 2.4.2:

**Theorem 2.4.3.** *Let  $G$  be a profinite group of rank  $d$  and let  $w$  be an outer-commutator word of degree  $n$ . Then*

$$\|w\| \leq F(d)^n,$$

where  $F(d)$  depends only on the rank of  $G$ , and may be taken to be  $2f(d) + d$ , where  $f(d)$  is as in Theorem 2.3.2.

If  $G$  is a pro-soluble group of rank  $d$  then  $F(d)$  may, in fact, be taken to be  $144d^2 + 93d$ . When  $G$  is pro-nilpotent we can make the bound much tighter:

**Theorem 2.4.4.** *Let  $G$  be a pro-nilpotent group of rank  $d$  and let  $w$  be an outer-commutator word of degree  $n$ . Then*

$$\|w(G)\| \leq (2d)^n.$$

In order to prove this we will use a technique from [25]; this is, in fact, the first step in the proof of Theorem 2.3.2. As in [25], we will employ the notation  $[H, {}_n G]$ , defined by

$$[H, {}_1 G] := [H, G] \quad \text{and} \quad [H, {}_{n+1} G] := [[H, {}_n G], G].$$

**Lemma 2.4.5.** *Let  $G$  be a finite group containing normal subgroups  $H = \langle h_1, \dots, h_{d_1} \rangle$  and  $K = \langle k_1, \dots, k_{d_2} \rangle$ . Then given any  $n \geq 0$ :*

$$[H, K] = [H, k_1] \cdots [H, k_{d_2}] [K, h_1] \cdots [K, h_{d_1}] [[H, K], {}_n G].$$

*Proof.* Without loss of generality we will assume that  $G = HK$ ; so that  $G = \langle h_1, \dots, h_{d_1}, k_1, \dots, k_{d_2} \rangle$ . Because  $G$  is finite, these are also semigroup generators for  $G$ . We will now proceed by induction on  $n$ . The result is trivial if  $n = 0$  and follows from Lemma 2.3.3 when  $n = 1$ . Now suppose that  $n \geq 2$  and that the result is true for smaller values of  $n$ .

Write  $A := [[H, K]_{n-2} G]$  and let  $g \in [H, K]$ . By hypothesis

$$g = [x_1, k_1] \cdots [x_{d_2}, k_{d_2}] [y_1, h_1] \cdots [y_{d_1}, h_{d_1}] \lambda,$$

with  $x_1, \dots, x_{d_2} \in H$ ,  $y_1, \dots, y_{d_1} \in K$ , and  $\lambda \in [A, G]$ . Modulo  $[[H, K]_n G]$  the subgroup  $[A, G]$  is central, so by Lemma 2.3.3:

$$\lambda = [a_1, k_1] \cdots [a_{d_2}, k_{d_2}] [b_1, h_1] \cdots [b_{d_1}, h_{d_1}] \mu,$$

with  $a_1, \dots, a_{d_2}, b_1, \dots, b_{d_1} \in A$  and  $\mu \in [[H, K]_n G]$ . We can view  $a_1, \dots, a_{d_2}$  as elements of  $H$  and  $b_1, \dots, b_{d_1}$  as elements of  $K$ . Each of the multiplicands above is central modulo  $[[H, K]_n G]$ , so in a similar way to Lemma 2.3.3 we can ‘collect’ the commutators and write

$$g = [a_1 x_1, k_1] \cdots [a_{d_2} x_{d_2}, k_{d_2}] [b_1 y_1, h_1] \cdots [b_{d_1} y_{d_1}, h_{d_1}] \mu',$$

with  $\mu' \in [[H, K]_n G]$ . This completes the induction.  $\square$

**Corollary 2.4.6.** *Let  $G$  be a pro-nilpotent group. Suppose that  $H \triangleleft_c G$  is a  $d_1$ -generator group and that  $K \triangleleft_c G$  is a  $d_2$ -generator group. Then the width of  $[H, K]$  is finite, and bounded by*

$$d_1 + d_2.$$

*Proof.* Without loss of generality let  $G = HK$ . If  $N \triangleleft_o G$  then  $G/N$  is a nilpotent group, and  $[H/N, K/N]$  has width  $d_1 + d_2$ . It follows, by Proposition 1.2.4, that the width of  $[H, K]$  is also bounded by  $d_1 + d_2$ .  $\square$

The arguments leading up to Theorem 2.4.3 now show that if  $w = [w_1 | w_2]$  is an independent commutator, and  $G$  is a pro-nilpotent group of rank  $d$ , then

$$\|w(G)\| \leq 2d\|w_1\|\|w_2\|.$$

A simple induction now establishes Theorem 2.4.4.

## 2.5 Notes

I believe that the counterexample  $w := [[x_1, x_2], [x_1, x_2, x_2]]$  described in section 2.2 is worthy of note because it requires only two variables, the smallest number of variables possible in a non-trivial commutator word. I would also consider this word to be ‘minimal’ amongst two-variable counterexamples. Of course we have no definition of the complexity of a commutator word with which to assess minimality, but Theorem 2.1.4 shows that all words which are based on a simple commutator (for example the *Engel* words, de-

finned inductively by  $v_1 := [x_1, x_2]$  and  $v_n := [v_{n-1}, x_2]$ ) have finite width in finitely generated pro- $p$  groups. The only other candidate for a ‘minimal’ counterexample is then  $[x_1, x_2]^p$ . I would argue that this is not as good as our counterexample because although  $[x_1, x_2]^p$  is a commutator word, it is a *product* of commutators whereas  $w$  is a single commutator.

Despite my above arguments, I am not suggesting that my counterexample is better than Roman’kov’s original counterexample  $[[x_1, x_2], [x_3, x_4]]$ . This word is not only an outer-commutator word, but a derived word. In its own way, this word is also a minimal counterexample, for all outer-commutator words of lower degree have finite width in finitely generated *profinite* groups.

The original result of Nikolov and Segal on which Lemma 2.4.5 is based states that in a  $d$ -generator pro-nilpotent group  $\| [H, G] \|$  is bounded by  $d$  for any  $H \triangleleft_c G$ . Therefore the main result of Corollary 2.4.6 (that  $[H, K]$  has finite width) could have been deduced without Lemma 2.4.5, by taking images in  $[H \cap K, G]$  and following the strategy of Theorem 2.3.5. This approach, however, gives a bound of  $2d_1 + d_2$ , which is always greater than that found in Corollary 2.4.6.

With this in mind, we might well ask whether the result of Theorem 2.3.5 could be obtained by putting Lemma 2.4.5 through the ‘machinery’ which Nikolov and Segal used to arrive at Theorem 2.3.2. The answer is *yes*, but this method yields a bound of  $f(d_1 + d_2)$ ; the non-linearity of  $f$  is more than enough to ensure that this value is greater than  $f(d_1) + f(d_2) + d_1$ .



# Chapter 3

## Polycyclic pro- $p$ groups

### 3.1 Introduction

The purpose of this chapter is to provide an embedding theorem which we will need to prove our main result in Chapter 4. We will state and prove this embedding theorem in Section 3.2, but we will first describe the main objects with which we will be dealing.

**Definition 3.1.1.** A pro- $p$  group  $G$  is *polycyclic* if there exists a finite subnormal chain of closed subgroups

$$1 = G_0 \triangleleft_c G_1 \triangleleft_c \cdots \triangleleft_c G_{n-1} \triangleleft_c G_n = G \quad (3.1)$$

such that each chain factor  $G_i/G_{i-1}$  is procyclic. That is, if each chain factor is a cyclic finite  $p$ -group or isomorphic to  $\mathbb{Z}_p$ .

A finite cyclic group has rank 1, as does  $\mathbb{Z}_p$ ; therefore a simple induction establishes that any polycyclic pro- $p$  group has finite rank. Under certain

conditions the converse is also true:

**Lemma 3.1.2.** *A pro- $p$  group of finite rank which satisfies a non-trivial group identity is polycyclic.*

*Proof.* Let  $G$  be a pro- $p$  group of finite rank which satisfies a non-trivial group identity. Pro- $p$  groups of finite rank are linear over  $\mathbb{Z}_p$ , and a theorem of Platonov states that a linear group is either virtually soluble or it generates the variety of all groups (see Theorem 10.15 of [38]); therefore  $G$  is virtually soluble and, in fact, soluble because it is pro- $p$ . But any soluble pro- $p$  group of finite rank is a polycyclic pro- $p$  group (see Proposition 8.2.2 of Wilson [41]).  $\square$

**Remark.** In the above proof, we could have deduced that  $G$  was virtually soluble using the well known Tit's Alternative, which states that a linear group is either virtually soluble or contains a non-Abelian free group. Platonov's result, however, is adequate for our result, and its proof is significantly less complicated.

The closed subgroups of a polycyclic pro- $p$  group  $G$  are also polycyclic, as are its quotients by closed normal subgroups. As with abstract polycyclic groups,  $G$  contains a unique maximal closed normal nilpotent subgroup

$$\text{Fitt}(G)$$

called the *Fitting Subgroup* of  $G$ , and the elements of  $\text{Fitt}(G)$  which have finite order form a finite subgroup.

**Lemma 3.1.3.** *If  $G$  is a polycyclic pro- $p$  group, and  $N := \text{Fitt}(G)$ , then  $G/N$  is Abelian-by-finite.*

A proof of this result can be found in Section 8.2 of Wilson [41]. It also follows from the Lie–Kolchin–Mal’cev Theorem which asserts that soluble linear groups are nilpotent-by-Abelian.

For any profinite group  $G$  which is virtually a pro- $p$  group of finite rank there is a unique non-negative integer  $\dim(G)$ , called the *dimension* of  $G$ . The reader is directed to Chapter 8 of Dixon et. al. [2] (and particularly Definition 8.37 therein) for specific details. What is important for our purposes is that for any closed normal subgroup  $N$  of  $G$ :

$$\dim(G) = \dim(N) + \dim(G/N),$$

and that  $\dim(N) = \dim(G)$  if and only if  $N$  is open in  $G$ .

In the case of polycyclic pro- $p$  groups the dimension coincides with the number of infinite cyclic factors in any ‘polycyclic series’ (3.1); that is, the dimension is the same as the *Hirsch Length* of the polycyclic group.

## 3.2 An embedding theorem

The main purpose of this chapter is to prove the following:

**Theorem 3.2.1.** *Let  $G$  be a profinite group which is virtually a polycyclic pro- $p$  group. Suppose that the Fitting subgroup  $N$  of  $G$  is torsion-free. Then  $G$  can be embedded as a subgroup of finite index in a profinite group  $G_1$  which*

has the form

$$G_1 = N_1G = N_1M_1,$$

where  $N_1$  is a finitely generated closed normal nilpotent subgroup of  $G_1$ , and a finite extension of  $N$ , and  $M_1$  is a finitely generated closed virtually nilpotent subgroup of  $G_1$ .

We will prove this via a technical result which uses ideas from the cohomology of groups. It is not the intention of this thesis to discuss group cohomology in any great detail, so we shall simply state the results which will be of most use to us. The reader is directed to Chapter 10 of Lennox and Robinson [19] for a thorough account of the theory behind these results. In the notes at the end of this chapter we will make it clear which of the results in [19] we have used.

Let  $G$  be a group and a  $T$  be a right  $\mathbb{Z}G$ -module. For our purposes we will define the 0th *homology group* to be

$$H_0(G, T) := T/T(G - 1),$$

and the 0th *cohomology group* to be

$$H^0(G, T) := C_T(G) := \{t \in T \mid t^g = t \text{ for all } g \in G\}.$$

We will also need to use the first and second cohomology groups  $H^1(G, T)$  and  $H^2(G, T)$  respectively. We do not need to know about the structure of these groups here; what is important are the following results:

**Proposition 3.2.2.** *All the complements of  $T$  in  $T \rtimes G$  are conjugate if and only if  $H^1(G, T) = 0$ .*

**Proposition 3.2.3.** *Every extension of  $T$  by  $G$  splits if and only if  $H^2(G, T) = 0$ .*

The key to the technical result which we will present is the following relationship between  $H_0(G, T)$  and the cohomology groups  $H^0(G, T)$ ,  $H^1(G, T)$ , and  $H^2(G, T)$ :

**Theorem 3.2.4** (See Robinson, [28]). *Suppose that  $G$  is a nilpotent group, that  $R$  is a ring with identity and that  $T$  is a Noetherian  $RG$ -module. Then*

$$H_0(G, T) = 0 \implies H^n(G, T) = 0 \text{ for all } n \geq 0.$$

**Definition 3.2.5.** Given a torsion-free nilpotent group  $N$  there exists a unique (up to isomorphism) minimal torsion-free nilpotent group, denoted  $N^{\mathbb{Q}}$ , which contains  $N$  and contains a  $k$ th root for each element of  $N$  and each  $k \in \mathbb{N}$ . This group is called the *Mal'cev completion* of  $N$ .

Although the Mal'cev completion is an important construction in group theory, it will play a purely technical role here, so we will only summarise the properties which are of use to us. For a more general description of the Mal'cev completion see Chapter 6 of [8], Part II of [2], Section 2 of [5], and Chapter 2 of [19].

The following will be particularly useful, and the details can be found in Section 2.1 of [19]:

**Lemma 3.2.6.** *Let  $N$  be a torsion-free nilpotent group and write  $V := N^{\mathbb{Q}}$ . Let  $Z$  be the centre of  $V$  and denote the centre of  $N$  by  $\zeta(N)$ . Then*

$$(i) \quad Z \cap N = \zeta(N);$$

(ii)  $V/Z$  is torsion-free, and the Mal'cev completion of  $N/\zeta(N)$ .

Given any  $k \in \mathbb{N}$  define the group  $N^{\frac{1}{k}}$  to be the subgroup of  $N^{\mathbb{Q}}$  generated by all the  $k$ th roots of the elements of  $N$ .

If  $N$  is a finitely generated torsion-free nilpotent profinite group then  $N$  has finite index in  $N^{\frac{1}{k}}$ ; one consequence of this is that the profinite topology of  $N$  induces a profinite topology on  $N^{\frac{1}{k}}$ , and another is that  $N^{\frac{1}{k}}$  is finitely generated. Thus  $N^{\frac{1}{k}}$  is also a finitely generated torsion-free nilpotent profinite group.

If  $G$  is a group which contains a torsion-free normal nilpotent subgroup  $N$  then it is possible to construct a group  $\Gamma$  which contains a copy of  $G$  and  $N^{\mathbb{Q}}$  (for details of this construction see, for example, [5]). If we identify  $G$  and  $N^{\mathbb{Q}}$  with their copies in  $\Gamma$  then

$$(i) \quad N^{\mathbb{Q}} \triangleleft \Gamma;$$

$$(ii) \quad N^{\mathbb{Q}} \cap G = N; \text{ and}$$

$$(iii) \quad \Gamma = N^{\mathbb{Q}}G.$$

As a consequence of these properties we also have

$$N^{\mathbb{Q}}G/N^{\mathbb{Q}} \cong G/N.$$

It is usual to denote the group  $\Gamma$  by  $N^{\mathbb{Q}}G$ , and in a similar way a group  $N^{\frac{1}{k}}G$  can be constructed for any  $k \in \mathbb{N}$ . In this case the key properties which we will need are:

- (i)  $N^{\frac{1}{k}} \triangleleft N^{\frac{1}{k}}G$ ;
- (ii)  $N^{\frac{1}{k}}G/N^{\frac{1}{k}} \cong G/N$ ; and
- (iii)  $|N^{\frac{1}{k}}G : G| < \infty$ .

We are now ready to prove a highly technical result which is the backbone of our embedding theorem. The main difference between this result and the actual embedding theorem is that much of the action here will not be taking place inside profinite groups. The main aim of the proof of the embedding theorem will be to take this result and show how the necessary groups can be replaced with profinite groups.

**Theorem 3.2.7.** *Let  $G$  be a finitely generated profinite group which is virtually a pro- $p$  group. Suppose that  $G$  contains a finitely generated torsion-free closed normal nilpotent subgroup  $N$  such that  $G/N$  is virtually nilpotent. Write  $\Gamma := N^{\mathbb{Q}}G$  and  $V := N^{\mathbb{Q}}$ . Then there exists a finitely generated virtually nilpotent profinite group  $M$  (which is a subgroup of  $\Gamma$ ) such that  $\Gamma = VM$ .*

*Proof.* We proceed by induction on  $\dim V$ . If  $\dim V = 0$  then  $N^{\mathbb{Q}} = \{1\}$  and we may then take  $M = G$ .

Now suppose that  $\dim V > 0$ . The centre  $Z$  of  $V$  is non-trivial, so it is infinite because  $V$  is torsion-free; therefore  $V/Z$  has strictly smaller dimension than  $V$ . Denote the centre of  $N$  by  $\zeta(N)$ ; then  $Z \cap G = \zeta(N)$ . The

quotient  $N/(Z \cap G)$  is therefore a finitely generated torsion-free closed normal subgroup of  $G/(Z \cap G)$ , and  $V/Z$  is the Mal'cev completion of  $N/(Z \cap G)$  by Lemma 3.2.6. We also have that

$$\Gamma/Z = (V/Z) \cdot (G/(Z \cap G)).$$

Therefore by hypothesis there exists a finitely generated virtually nilpotent profinite group  $L/Z$  (which is a subgroup of  $\Gamma/Z$ ) such that  $\Gamma = VL$ .

Now let  $D := V \cap L$ . Because  $L/Z$  is virtually nilpotent and  $D/Z$  is nilpotent there exists a normal subgroup  $L_1$  of finite index in  $L$  such that  $D \leq L_1 \leq L$  and  $L_1/Z$  is nilpotent.

Note that  $Z$  is the  $\mathbb{Q}_p$ -vector space spanned by the free  $\mathbb{Z}_p$ -module  $\zeta(N)$ . The action of  $G$  on  $\zeta(N)$  is continuous, and so  $G$  acts on  $\zeta(N)$  by  $\mathbb{Z}_p$ -linear automorphisms. These automorphisms can then be extended to  $\mathbb{Q}_p$ -linear automorphisms on  $Z$ . Because  $Z$  is the centre of  $N^{\mathbb{Q}}$  the elements of  $\Gamma$ , and in particular the elements of  $L_1$ , act on  $Z$  in the same way as  $G$ . Hence the action of  $L_1$  on  $Z$  is  $\mathbb{Q}_p$ -linear. Viewing the  $L_1$ -module structure of  $Z$  additively, we can define a descending chain of subgroups inductively by

$$Z_0 = Z, \quad \text{and} \quad Z_i = Z_{i-1}(L_1 - 1).$$

Because  $Z$  is finite dimensional there must exist some  $r \in \mathbb{N}$  such that  $Z_r = Z_{r+i}$  for all  $i \in \mathbb{N}$ . We will distinguish two cases:

**Case 1:**  $Z_r = \{0\}$ . In this case  $L_1$  is nilpotent, and so  $L$  is virtually nilpotent. The quotient  $L/D$  is isomorphic to  $G/N$ , and is a finitely gener-

ated profinite group. We now pick a finite topological generating set  $\bar{l}_1, \dots, \bar{l}_d$  for  $L/D$  and choose preimages  $l_1, \dots, l_d$  in  $L$ . Because there are only finitely many elements here there exists  $k \in \mathbb{N}$  such that  $l_1, \dots, l_d \in N^{\frac{1}{k}}G$ . Now  $N^{\frac{1}{k}}G$  is a profinite group, so take  $M$  to be the closure in  $N^{\frac{1}{k}}G$  of  $\langle l_1, \dots, l_d \rangle$ . For larger values of  $k$  the groups  $N^{\frac{1}{k}}G$  share the same topology, so  $N^{\frac{1}{k}}G$  induces the original topology on  $G$ , and  $M$  is well defined in this topology. Therefore  $L = DM$ , and

$$\Gamma = VL = VDH = VM,$$

where  $M$  is a finitely generated virtually nilpotent profinite group.

**Case 2:**  $Z_r > \{0\}$ . Write  $T := Z_r$ . We have  $T = T(L_1 - 1)$ , which is to say that  $H_0(L_1/T, T) = 0$ . Since  $T$  is a  $\mathbb{Q}_p L_1$ -module of finite  $\mathbb{Q}_p L_1$ -dimension it is certainly Noetherian; hence by Theorem 3.2.4:

$$H^0(L_1/T, T) = H^1(L_1/T, T) = H^2(L_1/T, T) = 0.$$

Because  $H^2(L_1/T, T) = 0$  it follows that every extension of  $T$  by  $L_1/T$  splits. In particular  $L_1$  is a split extension of  $T$ , so let  $K$  be a complement to  $T$  in  $L_1$ ; then  $L_1 = TK$  and  $T \cap K = 1$ . (Note that we are *not* claiming here that  $L_1$  is closed.)

Now take any  $x \in L$ . Because  $H^1(L_1/T, T) = 0$  we know that all of the complements for  $T$  in  $L_1$  are conjugate under the action of  $T$ . The group  $K^x$  is contained in  $L_1$  so there exists  $t \in T$  such that  $K^x = K^t$ . This means that  $x \in TN_L(K)$ . Writing  $H := N_L(K)$ , we then have  $L = TH$ .

It is clear that  $T \cap H \leq N_T(K)$ . Also

$$N_T(K) = C_T(K) = 1$$

because  $H^0(L_1/T, T) = 0$ . Therefore  $T \cap H = 1$  and

$$H \cong TH/T = L/T;$$

hence  $H$  is virtually nilpotent.

Because  $TH = L$  it is certainly the case that  $DH = L$ . Thus  $H/(H \cap D)$  is isomorphic to  $L/D$ , and hence to  $G/N$ . Thus  $H/(H \cap D)$  is a finitely generated profinite group. We can now apply the same argument as in Case 1 to find a finitely generated profinite subgroup  $M$  of  $H$  (closed in  $N^{\frac{1}{k}}G$ , for some  $k \in \mathbb{N}$ ) such that  $H = (H \cap D)M$ . We then have

$$\Gamma = VL = VTH = VT(H \cap D)M = VM,$$

as required. □

*Proof of Theorem 3.2.1.* Let  $G$  be a profinite group which is virtually a polycyclic pro- $p$  group, and suppose that the Fitting subgroup  $N$  of  $G$  is torsion-free. By Lemma 3.1.3 the quotient  $G/N$  is virtually Abelian; therefore the hypotheses of Theorem 3.2.7 are satisfied. This means that there exists a finitely generated virtually nilpotent profinite subgroup  $M$  of  $N^{\mathbb{Q}}G$  such that

$$N^{\mathbb{Q}}G = N^{\mathbb{Q}}M.$$

Because  $M$  is finitely generated there exists  $k_1 \in \mathbb{N}$  such that  $M \leq N^{\frac{1}{k_1}}G$ , and because  $G$  is finitely generated there exists  $k_2 \in \mathbb{N}$  such that  $G \leq N^{\frac{1}{k_2}}M$ . Setting  $k := k_1k_2$ , we then have

$$N^{\frac{1}{k}}G = N^{\frac{1}{k}}M.$$

Now let  $N_1 := N^{\frac{1}{k}}$  and  $G_1 := N_1G$ . Because  $|N_1G : G|$  is finite the profinite topology of  $G$  induces a profinite topology on  $N_1G$ ; so  $G_1$  is a profinite group which is virtually a polycyclic pro- $p$  group (for the open polycyclic pro- $p$  subgroup of  $G$  is also open in  $G_1$ ). Now  $M$  is closed in  $G_1$ , so we can take  $M_1 := M$ . Then

$$G_1 = N_1G = N_1M_1,$$

where  $M_1$  is a finitely generated closed virtually nilpotent subgroup of  $G_1$ .  $\square$

### 3.3 Notes

In [29] Roman'kov shows that a polycyclic group  $G$  with torsion-free Fitting subgroup  $N$  can be embedded as a subgroup of finite index in a group  $G_1$  of the form

$$G_1 = N_1G = N_1M_1, \tag{3.2}$$

where  $N_1$  is a finite extension of  $N$  in its Mal'cev completion  $N^{\mathbb{Q}}$ , and  $M_1$  is a finitely generated virtually nilpotent group. This result is obtained via a clever, but complicated, proof which takes as its starting point a Zaičev decomposition for  $G$  (that is, a nilpotent subgroup  $M$  such that  $NM$  has finite index in  $G$ ) and then extends  $N$  and  $M$  in  $N^{\mathbb{Q}}G$ .

The arguments presented in this chapter provide an alternative proof for this result. We needed to consider profinite groups in Theorem 3.2.7 because our groups in Theorem 3.2.1 were profinite. However, dropping the concerns over topology, the arguments of Theorem 3.2.7 can be used to prove:

**Theorem.** *Suppose that  $\Gamma$  is a group containing a normal subgroup  $V := N^{\mathbb{Q}}$ , where  $N$  is a finitely generated torsion-free nilpotent group, and that  $\Gamma/V$  is finitely generated and virtually nilpotent. Then there exists a finitely generated virtually nilpotent subgroup  $M$  such that  $\Gamma = VM$ .*

If  $G$  is a virtually polycyclic group with torsion-free Fitting subgroup  $N$  then we can apply this result to  $N^{\mathbb{Q}}G$ . The arguments used to prove Theorem 3.2.1 can then be used to obtain the embedding theorem (3.2) above.

As promised, we will now detail the specific results in Lennox and Robinson [19] which we used to arrive at the cohomological results of this chapter:

That the 0th homology and cohomology groups,  $H_0(G, T)$  and  $H^0(G, T)$ , have the structure that we described is detailed on page 196 of [19]. The structure of  $H^1(G, T)$  is described in 10.1.7. What we have stated as Proposition 3.2.2 is Corollary 10.1.9 of [19], and what we have stated as Proposition 3.2.3 can be found as Corollary 10.1.14. The key result Theorem 3.2.4, describing the relationship between these (co)homology groups, is 10.3.1 in [19].

# Chapter 4

## Verbal subgroups of finite rank pro- $p$ groups

### 4.1 Introduction

Towards the end of Chapter 1 we gave some examples showing how the additional assumption of *finite rank* can simplify the study of verbal subgroups of profinite groups. We also used the assumption of finite rank in Chapter 2: firstly to enable an induction argument in our proof that the outer-commutator words have finite width (in a profinite group of finite rank), and secondly to provide a bound for that width. We will now shift our focus specifically to pro- $p$  groups of finite rank.

The main aim of this chapter is to present a proof that every profinite group which is virtually a pro- $p$  group of finite rank is verbally-elliptic. This result is not new, having been recently proved by Jaikin-Zapirain [14]. Jaikin-Zapirain's clever proof relies on analytical methods, using ideas from  $p$ -adic

differential geometry. In this chapter we will provide an alternative proof, relying entirely on group-theoretical ideas. Our strategy relies on the following key observation:

**Lemma 4.1.1.** *Every profinite group which is virtually a pro- $p$  group of finite rank is verbally-elliptic if and only if every profinite group which is virtually a polycyclic pro- $p$  group is verbally-elliptic.*

*Proof.* The forward implication is obvious. For the converse, let  $G$  be a profinite group which contains an open normal pro- $p$  group  $P$  of rank  $d$ . Let  $w$  be any word, and denote by  $W$  the closure in  $G$  of the verbal subgroup  $w(P)$ . By Lemma 1.1.10:

$$\|w(G)\| \leq \|w(G/[W, W])\| + \|w(G) \cap [W, W]\|_{w\{G\}}.$$

Our task will be to show that each of the terms on the right hand side is finite.

Define  $\tilde{G} := G/[W, W]$  and  $\tilde{P} := P/[W, W]$ . It is clear that  $\tilde{P}$  satisfies the law  $[w | w]$ , so by Lemma 3.1.2  $\tilde{P}$  is polycyclic. Hence  $\tilde{G}$  is virtually a polycyclic pro- $p$  group. Thus  $\|w(\tilde{G})\|$  is finite by hypothesis.

Because  $w\{P\}$  is a (topological) generating set for  $W$ , and  $P$  has finite rank  $d$ , there exist elements  $v_1, \dots, v_d \in w\{P\}$  such that

$$W = \overline{\langle v_1, \dots, v_d \rangle}.$$

By Proposition 2.3.1 every element of  $[W, W]$  has the form

$$[v_1, g_1][v_2, g_2] \cdots [v_d, g_d],$$

for some  $g_1, \dots, g_d \in W$ , and each commutator

$$[v_i, g_i] = v_i^{-1} v_i^{g_i}$$

has width 2 with respect to  $w\{P\}$ ; therefore  $[W, W] \leq w(P) \leq w(G)$ , and

$$\|[W, W] \cap w(G)\|_{w\{G\}} \leq 2d.$$

It now follows that the width of  $w(G)$  is finite. □

**Remark.** In the special case when  $G$  is a pro- $p$  of finite rank, the verbal subgroup  $w(\tilde{G})$  is Abelian. The reader may wonder whether this alone would be sufficient to guarantee that  $w(G)$  has finite width. Recall, however, that Ivanov's counterexample, discussed in Section 1.1, provides us with a verbal subgroup of infinite width which is *cyclic*.

In [29] Roman'kov shows that all polycyclic groups are verbally-elliptic. There is no obvious way to extract any bounds for the width from Roman'kov's work, so this result can not be used to directly imply the corresponding result for polycyclic pro- $p$  groups (via Proposition 1.2.4). Instead, we will prove this latter result from first principles, by following Roman'kov's strategy, and adapting his methods to pro- $p$  groups.

## 4.2 Generalised words

The starting point for our proof is the embedding theorem considered in the previous chapter. In order to use this we need quite a few background results. The driving force behind these are *generalised words*. These were first explicitly introduced by Roman'kov, who called them  $\Phi$ -words; although some of the ideas behind their use were implicit in the work of Turner-Smith [36, 35].

### 4.2.1 Definition and elementary properties

**Definition 4.2.1.** Let  $G$  be a group. A *generalised word* for  $G$ , in  $n$  variables, is an expression of the form

$$(x_{i_1}^{\epsilon_1})^{\phi_1} (x_{i_2}^{\epsilon_2})^{\phi_2} \cdots (x_{i_s}^{\epsilon_s})^{\phi_s}, \quad (4.1)$$

where  $s \in \mathbb{N}_0$  and for each  $j = 1, \dots, s$  we have  $i_j \in \{1, \dots, n\}$ ,  $\epsilon_j = \pm 1$ , and  $\phi_j \in \text{Aut } G$ .

As with our usual words, we will view a generalised word in  $n$  variables as a function  $G^{(n)} \rightarrow G$ , where the image of  $(g_1, \dots, g_n)$  is obtained by replacing each  $x_i$  by  $g_i$ , for  $i = 1, \dots, n$ . Note that only  $x_1, \dots, x_n$  are considered to be variables of the generalised word: the elements of  $\text{Aut } G$  which occur are *fixed* and are not replaced when we take values of  $w$  in  $G$ . Therefore we will write  $w(x_1, \dots, x_n)$  to denote a generalised word in  $n$  variables. Note that it is possible that the same variable may occur at different points in a generalised word with different automorphisms; so if  $v(x_1, \dots, x_n)$  is a generalised word

it is *not* necessarily the case that

$$v(x_1, \dots, x_n) = w(x_1^{\phi_1}, \dots, x_n^{\phi_n}),$$

for some (abstract) word  $w$ . We will define the vocabulary  $w\{G\}$  and the verbal subgroup  $w(G)$  of a generalised word  $w$  analogously to that of a usual word.

The definition we have given for a generalised word requires a group to be specified first. There is an alternative: we could define a generalised word in  $n$  variables to be an expression of the form (4.1) in which the elements  $\phi_j$  come from some arbitrary set  $\Phi$ . To obtain a value for such a word in a group  $G$  we would need to first assign the elements of  $\Phi$  to automorphisms of  $G$ . From this point of view, the generalised words which we defined in Definition 4.2.1 would then be ‘instances’ of these symbolically abstract generalised words.

The main purpose of this thesis is to explore properties of words, and we will use generalised words to do this. From this approach, to view a generalised word in the abstract sense described above feels somewhat clumsy, and disguises some of the difficulties which need to be faced when dealing with, for example, the images of verbal subgroups of generalised words. Because the topic of generalised words is relatively unexplored, however, it will be necessary for us to prove some quite technical results *about* generalised words; as we might expect, the more abstract approach has advantages in this case. Consequently we will be using both approaches in this work. We will take Definition 4.2.1 to be the definition of a generalised word, but remain aware that the automorphisms which appear could just be taken to be

arbitrary symbols, ready to be assigned to automorphisms of a group.

For the remainder of this section we shall examine the elementary properties of the verbal subgroups of generalised words. By our usual convention, we will define the width of a generalised word  $w$  in  $G$  to be the width of  $w(G)$  with respect to  $w\{G\}$ .

**Lemma 4.2.2.** *Every generalised word has width 1 in an Abelian group.*

*Proof.* Let  $A$  be any Abelian group and let  $w$  be a generalised word for  $A$ . There exists a word  $v$  (not generalised) such that

$$w(x_1, \dots, x_n) = v(x_{i_1}^{\phi_1}, \dots, x_{i_m}^{\phi_m}),$$

where  $i_1, \dots, i_m \in \{1, \dots, n\}$  and  $\phi_1, \dots, \phi_m \in \text{Aut } A$ . Given any elements  $a_1, \dots, a_n, b_1, \dots, b_n \in A$  we have

$$\begin{aligned} w(a_1, \dots, a_n)w(b_1, \dots, b_n) &= v(a_{i_1}^{\phi_1}, \dots, a_{i_m}^{\phi_m})v(b_{i_1}^{\phi_1}, \dots, b_{i_m}^{\phi_m}) \\ &= v(a_{i_1}^{\phi_1}b_{i_1}^{\phi_1}, \dots, a_{i_m}^{\phi_m}b_{i_m}^{\phi_m}) \\ &= v((a_{i_1}b_{i_1})^{\phi_1}, \dots, (a_{i_m}b_{i_m})^{\phi_m}) \\ &= w(a_1b_1, \dots, a_nb_n), \end{aligned}$$

where the second equality follows from the proof of Lemma 1.1.8. It follows that  $w\{G\}$  is closed under multiplication and that

$$w(a_1, \dots, a_n)^{-1} = w(a_1^{-1}, \dots, a_n^{-1}).$$

Hence  $w\{A\}$  is a group, and  $w(A)$  has width 1. □

Clearly every word is also a generalised word. As we would expect, many of the elementary properties of words discussed in Chapter 1 also hold (in some form) for generalised words. We cannot, however, assume that all the properties of words automatically hold for generalised words. For example it is not known whether a verbal subgroup of a generalised word is necessarily normal.

This particular problem with verbal subgroups of generalised words is the failure of Lemma 1.1.3 to fully generalise to generalised words. If  $w$  is a generalised word for a group  $G$  and  $\theta : G \rightarrow H$  is any homomorphism then there is not, in general, a natural way to map the automorphisms of  $G$  which appear in  $w$  to automorphisms of  $H$ .

On the other hand, if  $\theta : G \rightarrow H$  is an *isomorphism* and  $w$  is a generalised word for  $G$ , then we can define  $w^\theta$  to be the generalised word for  $H$  obtained by replacing each automorphism  $\phi$  of  $G$  by the automorphism  $\theta^{-1}\phi\theta$  of  $H$ . Then

$$w(G)^\theta = w^\theta(G^\theta);$$

so a verbal subgroup of a generalised word remains a verbal subgroup under an isomorphism. This is a satisfying property (or rather, it would be most inconvenient if this wasn't true), but in the context of the width of verbal subgroups it has little application.

We really need to be able to drop the requirement that  $\theta : G \rightarrow H$  is an isomorphism. To do this we will need to impose some additional restrictions on  $\theta$  to ensure that the automorphisms which occur in a generalised word  $w$  can be mapped to automorphisms of  $H$ . One way to do this is to require

that  $\theta$  extends to a homomorphism of the holomorph  $G \rtimes \text{Aut } G$ . In practice mapping to the entire holomorph can be restrictive, so we will take a slightly different approach:

**Definition 4.2.3.** If  $E$  is an extension of a group  $G$  then we will say that a generalised word  $w$  for  $G$  takes *fixed values* in  $E$  if every automorphism which occurs in  $w$  is the restriction to  $G$  of an inner automorphism of  $E$ . Given such an extension, we will refer to the elements which give rise to the automorphisms in  $w$  as the *fixed values* of  $w$ .

This gives us a third way of thinking about generalised words, because every generalised word for every group  $G$  can be considered to take fixed values in some extension: namely the holomorph  $G \rtimes \text{Aut } G$ .

We can now state a result about homomorphisms quite easily:

**Lemma 4.2.4.** *Let  $E$  be an extension of a group  $G$  and let  $w$  be a generalised word for  $G$  which takes fixed values in  $E$ . If  $\theta : E \rightarrow F$  is a homomorphism then*

$$w(g_1, \dots, g_n)^\theta = w^\theta(g_1^\theta, \dots, g_n^\theta),$$

where  $w^\theta$  is the generalised word for  $G^\theta$ , taking fixed values in  $F$ , that is obtained from  $w$  by replacing each fixed value with its image under  $\theta$ . Therefore

$$w(G)^\theta = w^\theta(G^\theta).$$

In particular if  $K$  is a normal subgroup of  $E$  contained in  $G$  then

$$w(G)K/K = \tilde{w}(G/K),$$

where  $\tilde{w}$  is the generalised word obtained by replacing each fixed value in  $E$  with its natural image in  $E/K$ .

As a matter of notational convenience, when we are discussing the verbal subgroups of quotient groups we will omit the tilde over the  $w$  and write  $w(G/K)$ . Thus, as before, this denotes both the appropriate verbal subgroup of  $G/K$ , and the image of  $w(G)$  in  $G/K$ . As an immediate consequence of Lemma 1.1.7 we then have:

**Lemma 4.2.5.** *Let  $E$  be an extension of a group  $G$  and let  $w$  be a generalised word for  $G$  taking fixed values in  $E$ . If  $K$  is a normal subgroup of  $E$  which is contained in  $G$  then*

$$\|w(G)\| \leq \|w(G/K)\| + \|w(G) \cap K\|_{w\{G\}}.$$

If  $v$  and  $w$  are generalised words then we can define the independent product  $v \diamond w$  in exactly the same way as in Definition 1.1.13. The arguments which precede Lemma 1.1.14 also hold without adjustment to give:

**Lemma 4.2.6.** *Let  $W = \{w_1, \dots, w_k\}$  be a finite set of generalised words. Then there exists a single generalised word  $w = w_1 \diamond w_2 \diamond \dots \diamond w_k$  such that  $W(G) = w(G)$ , for all groups  $G$ . Furthermore, in any group*

$$\|w\| \leq \|w_1\| + \dots + \|w_k\|.$$

We will see some examples of generalised words in action in Section 4.3. For now we will illustrate their usefulness with an interesting application that will be useful later:

**Proposition 4.2.7.** *Let  $G$  be a group in which every generalised word has finite width. If  $E$  is a finite extension of  $G$  and  $w(x_1, \dots, x_n)$  is a (usual) word that is robust on every homomorphic image of  $E$ , then  $w$  has finite width in  $E$ .*

*Proof.* Let  $t_1, \dots, t_s$  be representatives for the cosets of  $G$  in  $E$  and consider the generalised words defined by

$$v_{i_1, \dots, i_n}(x_1, \dots, x_n) := w(t_{i_1}x_1, \dots, t_{i_n}x_n)w(t_{i_1}, \dots, t_{i_n})^{-1}.$$

The fixed values here are expressions in  $t_{i_1}, \dots, t_{i_n}$ , and the resulting generalised words are parametrised by the values of  $i_1, \dots, i_n$ . Let  $V$  be the set of all  $s^n$  such generalised words. Because  $V$  is a finite set there exists a single generalised word  $v$  such that  $v(G) = V(G)$ ; therefore  $V(G)$  has finite width. Each element of  $V\{G\}$  is a product of two elements of  $w\{E\}$ ; hence  $V(G)$  has finite width with respect to  $w\{E\}$ .

Now write  $N = V(G)$ . We claim that  $N$  is a normal subgroup of  $E$ : Given  $e \in E$  and  $i \in \{1, \dots, s\}$ , let  $ie \in \{1, \dots, s\}$  be such that  $t_{ie}$  is the representative of the coset containing the element  $t_i^e$ . Additionally, let  $h_{i,e} \in G$  be such that  $t_i^e = t_{ie}h_{i,e}$ . Then given  $g_1, \dots, g_n \in G$  we have

$$\begin{aligned} v_{i_1, \dots, i_n}(g_1, \dots, g_n)^e &= w(t_{i_1}^e g_1^e, \dots, t_{i_n}^e g_n^e)w(t_{i_1}^e, \dots, t_{i_n}^e)^{-1} \\ &= w(t_{i_1 e} h_{i_1, e} g_1^e, \dots, t_{i_n e} h_{i_n, e} g_n^e)w(t_{i_1 e} h_{i_1, e}, \dots, t_{i_n e} h_{i_n, e})^{-1} \\ &= w(t_{i_1 e} h_{i_1, e} g_1^e, \dots, t_{i_n e} h_{i_n, e} g_n^e)w(t_{i_1 e}, \dots, t_{i_n e})^{-1} \\ &\quad w(t_{i_1 e}, \dots, t_{i_n e})w(t_{i_1 e} h_{i_1, e}, \dots, t_{i_n e} h_{i_n, e})^{-1} \\ &= v_{i_1 e, \dots, i_n e}(h_{i_1, e} g_1^e, \dots, h_{i_n, e} g_n^e)v_{i_1 e, \dots, i_n e}(h_{i_1, e}, \dots, h_{i_n, e})^{-1}, \end{aligned}$$

which is an element of  $V(G)$ ; hence  $N$  is normal in  $E$ .

Now observe that the quotient  $G/N$  is marginal for  $w$  in  $E/N$ . We have that  $|E/N : G/N| = |E : G|$ , which is finite. Because  $w$  is robust on  $E/N$ , the verbal subgroup  $w(E/N)$  is finite. The result now follows from Lemma 1.1.10, noting that  $w(E) \cap N = V(G)$ .  $\square$

## 4.2.2 Application to nilpotent groups

We are yet to prove the existence of any infinite non-Abelian groups in which every generalised word has finite width. We will now prove that finitely generated nilpotent groups have this property.<sup>1</sup>

The main step in Roman'kov's proof of this result depends on the following number theoretical result:

**Theorem 4.2.8.** *For each positive integer  $c$  there exists a number  $g(c)$  which has the property that for every positive integer  $m$  there exist integers  $m_1, m_2, \dots, m_{g(c)}$  such that*

$$m = \pm m_1^c \pm m_2^c \pm \dots \pm m_{g(c)}^c.$$

This result is proved by Hardy and Wright as Theorem 401 in [9], and is a special case of Waring's Theorem, which was proposed by Edward Waring [37] in 1770 and proved by David Hilbert [11] in 1909. Waring's Theorem differs from the above result in that each  $\pm$  sign is replaced with a  $+$  sign.

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<sup>1</sup>In Section 4.4 we will see that we can weaken this hypothesis to nilpotent groups which need not be finitely generated, but in which every finitely generated subgroup can be generated by at most  $d$  elements, for some fixed value  $d$ .

For our purposes there is no problem with the negative terms, and the proof of Theorem 4.2.8 is much simpler than that of Waring's Theorem.

**Lemma 4.2.9** (Roman'kov). *Let  $L$  be a nilpotent group of class  $c$ , generated by a finite set  $A$ . Suppose that for each  $m \in \mathbb{N}$  the map  $\xi_m$ , taking each  $a \in A$  to the element  $a^m$ , extends to an endomorphism of  $L$ . If  $Y \subseteq L$  has the property that  $\xi_m(Y) \subseteq Y$  for each  $m$ , and  $H = \langle Y \rangle$ , then  $\|H \cap \gamma_c(L)\|_Y$  is finite.*

*Proof.* Suppose that  $L$  is generated by  $a_1, \dots, a_d$ . We can choose a generating set for  $\gamma_c(L)$  in which any generator  $b$  has the form

$$b = [a_{i_1}, \dots, a_{i_c}]$$

for some  $i_1, \dots, i_c \in \{1, \dots, d\}$ . This commutator is linear in each of its variables so, for each  $m \in \mathbb{N}$ ,

$$\xi_m(b) = [a_{i_1}^m, \dots, a_{i_c}^m] = [a_{i_1}, \dots, a_{i_c}]^{m^c} = b^{m^c}.$$

Because  $\gamma_c(L)$  is Abelian this property holds for all the elements of  $\gamma_c(L)$ .

Now let  $g := g(c)$  be a natural number such that for all  $m \in \mathbb{N}$  there exist  $m_1, \dots, m_g$  with  $m = \pm m_1^c \pm \dots \pm m_g^c$ . If  $k \in \gamma_c(L)$  then

$$k^m = k^{\pm m_1^c} \dots k^{\pm m_g^c} = \xi_{m_1}(k)^{\pm 1} \dots \xi_{m_g}(k)^{\pm 1}. \quad (4.2)$$

Because  $\xi_m(Y) \subseteq Y$  we know that  $\|\xi_m(h)\|_Y \leq \|h\|_Y$  for all  $h \in H$ . Also  $\|h^{-1}\|_Y = \|h\|_Y$ . Therefore, by Equation (4.2) it follows that if  $z \in H \cap \gamma_c(L)$

and  $m \in \mathbb{Z}$  then

$$\|z^m\|_Y \leq g(c) \times \|z\|_Y.$$

Notice that this bound does not depend on  $m$ .

If  $z_1, \dots, z_t$  are generators of  $H \cap \gamma_c(L)$  then every element of  $H \cap \gamma_c(L)$  has the form

$$z_1^{m_1} \cdots z_t^{m_t},$$

for some  $m_1, \dots, m_t \in \mathbb{Z}$ , and from the above:

$$\|z_1^{m_1} \cdots z_t^{m_t}\|_Y \leq g \times (\|z_1\|_Y + \cdots + \|z_t\|_Y).$$

Hence  $\|H \cap \gamma_c(L)\|_Y$  is finite. □

That finitely generated nilpotent groups are verbally elliptic follows easily from this result:

**Corollary 4.2.10** (Roman'kov). *Every finitely generated nilpotent group is verbally elliptic.*

*Proof.* We will first prove that every finitely generated free nilpotent group is verbally-elliptic. We will proceed by induction on the nilpotency class of such a group, with the base case following from Lemma 1.1.8.

Suppose that  $L$  is a finitely generated free nilpotent group of class  $c$ , and that the result is true for finitely generated free nilpotent groups of smaller class. By Lemma 1.1.10, for any word  $v$ :

$$\|v(L)\| \leq \|v(L/\gamma_c(L))\| + \|v(L) \cap \gamma_c(L)\|_{v\{L\}}.$$

By induction  $v(L/\gamma_c(L))$  has finite width, and taking  $Y$  to be  $v\{L\}$  in Lemma 4.2.9 it follows that  $\|v(L) \cap \gamma_c(L)\|_{v\{L\}}$  is also finite. Hence  $v(L)$  has finite width. This completes the claim that finitely generated free nilpotent groups are verbally-elliptic.

Now let  $w$  be any word, and let  $G$  be any finitely generated nilpotent group. There exists a surjective homomorphism  $\theta : L \rightarrow G$  for some finitely generated free nilpotent group  $L$ , and by Lemma 1.1.9

$$\|w(G)\| \leq \|w(L)\|.$$

It follows that  $w(G)$  has finite width. □

To adapt these techniques to generalised words we need to be more careful: as discussed earlier there are difficulties surrounding generalised words and endomorphisms, and Lemma 1.1.9 need not automatically apply. In particular this lemma might not apply when we take images from the free group. Therefore we will work in a particular subgroup of a free group, and use the symbolic representations of generalised words which we discussed in the previous section. We make the following definitions:

**Definition 4.2.11.** Let  $A$  and  $\Psi$  be sets, and let

$$T = \{a, a^\psi \mid a \in A, \psi \in \Psi\}.$$

The *free group for generalised words* on  $A$  and  $\Psi$  is the subgroup of the free group on  $A \cup \Psi$  generated by  $T$ . We will denote this group by  $FG(A, \Psi)$ .

If  $v(x_1, \dots, x_n)$  is a generalised word taking fixed values in  $\Psi$ , we now

define the *restricted vocabulary* of  $v$  to be

$$v_r\{FG(A, \Psi)\} := \{v(k_1, \dots, k_n) \mid k_1, \dots, k_n \in F(A)\};$$

the restricted verbal subgroup of  $v$  is then the subgroup generated by this set.

**Lemma 4.2.12.** *Let  $A$ ,  $\Psi$ , and  $T$  be as above. Then the subgroup  $FG(A, \Psi)$  is freely generated by  $T$ .*

*Proof.* Without loss of generality we may assume that  $A$  and  $\Psi$  are finite. Let  $F$  be the free group on  $A \cup \Psi$ , let  $V$  be an elementary Abelian 2-group of rank  $|A|$  and let  $W = V \wr \langle \Psi \rangle$ . Now define a homomorphism  $\theta : F \rightarrow W$  by sending the elements of  $A$  to a basis of  $V$ , and the elements of  $\Psi$  to their copy in  $\langle \Psi \rangle$ . The image of  $FG(A, \Psi)$  under this homomorphism is then the direct product of the cosets  $V$  and  $V\psi$  for  $\psi \in \Psi$ . There are  $|\Psi| + 1$  such cosets so the image of  $FG(A, \Psi)$  has rank

$$|A| \times (|\Psi| + 1) = |T|.$$

This means that the Abelianisation of  $FG(A, \Psi)$  must have rank at least  $|T|$ . But  $FG(A, \Psi)$  is a subgroup of a free group so it must be free by the Nielsen–Schreier Theorem. Hence  $FG(A, \Psi)$  must be freely generated by  $T$ , or it would have smaller rank.  $\square$

So how will we use our free group for generalised words? Well if we have a group  $G$  and a generalised word  $w$ , we will choose  $A$  so that it can map onto a generating set of  $G$ , and  $\Psi$  so that it can map on to the automorphisms of

$G$  which appear in  $w$ ; then the free group on  $A \cup \Psi$  can be made to map on to a suitable extension of  $G$  (and  $w$  will take fixed values from this extension). Under such a mapping,  $G$  will be the image  $FG(A, \Psi)$  and  $w(G)$  will be the image of  $v_r(FG(A, \Psi))$ . We will then be able to bound the width of  $w(G)$  by the width of  $v_r(FG(A, \Psi))$ . Note that if  $A$  and  $\Psi$  are finite then  $FG(A, \Psi)$  is finitely generated.

We will be applying the arguments of Lemma's 4.2.9 and 4.2.9 to our free group for generalised words. This requires us to take quotients of this group by subgroups which are *not* invariant under conjugation by the elements of  $\Psi$ ; this presents a problem because a generalised word may turn out to not be a generalised word for such a quotient. Fortunately this will prove to not matter, because we only need to deal with the restricted vocabularies:

**Lemma 4.2.13.** *Write  $K := FG(A, \Psi)$  and let  $v(x_1, \dots, x_n)$  be a generalised word taking fixed values in  $\Psi$ . Suppose that  $N$  is a normal subgroup of  $K$  with the property that  $n^\psi \in N$  whenever  $n \in N \cap F(A)$  and  $\psi \in \Psi$ . For  $k \in F(A)$  and  $\psi \in \Psi$  define*

$$(kN)^\psi = k^\psi N.$$

*Then the values of  $v_r$  in  $K/N$  are well defined. Additionally,  $v_r\{K/N\}$  and  $v_r(K/N)$  are the images in  $K/N$  of  $v_r\{K\}$  and  $v_r(K)$  respectively.*

*Proof.* Let  $k_1, k_2 \in K$  be such that  $k_1 N = k_2 N$ , and let  $\psi \in \Psi$ . Then

$$k_1^\psi (k_2^\psi)^{-1} = k_1 (k_2^{-1})^\psi = (k_1 k_2^{-1})^\psi \in N;$$

hence  $(k_1 N)^\psi = (k_2 N)^\psi$ . □

This is not a sophisticated result, but will prove to be very useful because verbal subgroups of  $FG(A, \Psi)$  have the property required by the lemma; thus we may safely take quotients by verbal subgroups. Because  $v_r(K/N)$  is the image in  $K/N$  of  $v_r(K)$  the arguments leading to Lemma 1.1.10 and Lemma 4.2.5 can be easily used to show the following:

**Lemma 4.2.14.** *Let  $A$  and  $\Psi$  be sets. Let  $v$  be a generalised word taking fixed values in  $\Psi$ , and let  $K := FG(A, \Psi)$  be the free group for generalised words. If  $N$  is a normal subgroup satisfying the condition of Lemma 4.2.13 then*

$$\|v_r(K)\| \leq \|v_r(K/N)\| + \|v_r(K) \cap N\|_{v_r\{K\}}.$$

We are now ready to adapt the strategy of Corollary 4.2.10 to generalised words:

**Theorem 4.2.15** (Roman'kov). *Every generalised word has finite width in a finitely generated nilpotent group.*

*Proof.* Let  $A$  and  $\Psi$  be finite sets, let  $K := FG(A, \Psi)$ , and let  $v$  be a generalised word taking fixed values in  $\Psi$ . We claim that  $v_r$  has finite width in  $K/\gamma_{c+1}(K)$  for every  $c \in \mathbb{N}$ . The proof is by induction on  $c$ , with the base case following from Lemma 4.2.2.

Suppose that  $v_r$  has finite width in  $K/\gamma_c(K)$  and let  $L = K/\gamma_{c+1}(K)$ ; thus  $L$  is a finitely generated free nilpotent group. Additionally,

$$L/\gamma_c(L) \cong K/\gamma_c(K),$$

so the width of  $v_r(L/\gamma_c(L))$  is finite by hypothesis. In order to show that

$v_r(L)$  has finite width it will now be enough to show that  $v_r(L) \cap \gamma_c(L)$  has finite width with respect to  $v_r\{L\}$ , and apply Lemma 4.2.14.

To do this we consider the mappings  $\xi_m$  which take each of the free generators of  $L$  to its  $m$ th power. These extend to endomorphisms of  $L$ , and  $\xi_m(v_r\{L\}) \subseteq v_r\{L\}$ . Therefore, by 4.2.9,  $v_r(L) \cap \gamma_c(L)$  has finite width with respect to  $v_r\{L\}$ . Thus  $v_r(L)$  has finite width, and this completes the induction.

What now remains is for us to show how the objects in the previous paragraphs can be mapped on to a generalised verbal subgroup of any finitely generated nilpotent group: Suppose that  $G$  is such a group, of class  $c$ , and let  $w(x_1, \dots, x_n)$  be a generalised word for  $G$ . Choose a finite generating set  $B$  for  $G$  and let  $\Phi$  be the set of automorphisms of  $G$  which appear in  $w$ ; note that  $\Phi$  is finite. Choose  $A$  and  $\Psi$  to have the same size as  $B$  and  $\Phi$  respectively; there then exists an epimorphism  $\theta : K \rightarrow G$  (restricted from a bijection between the elements of  $A$  and  $B$  and elements of  $\Psi$  and  $\Phi$ ). Let  $v$  be the symbolic generalised word obtained from  $w$  by replacing each  $\phi \in \Phi$  by its preimage in  $\Psi$ . Because  $G$  is nilpotent of class  $c$  we have  $\gamma_{c+1}(K) \leq \ker \theta$ , and thus  $\theta$  induces an epimorphism  $L \rightarrow G$ , where  $L := K/\gamma_{c+1}(K)$ . The images of  $v_r\{L\}$  and  $v_r(L)$  under this epimorphism are  $w\{G\}$  and  $w(G)$  respectively, and  $v_r(L)$  has finite width; hence  $w(G)$  has finite width.  $\square$

**Remark.** In the above proof the width of  $w(G)$  was bounded by the width of  $v_r(L)$ . Here  $v$  is the symbolic representation of  $w$ , and  $L$  depends only on the nilpotency class  $c$  and the number of generators  $d$  of  $G$ . Therefore the width of  $w(G)$  is bounded by a function depending only on  $c$ ,  $d$ , and  $w$ . This

fact will be critical in our work which follows.

Notice also that the induction hypothesis made reference to *restricted* verbal subgroups of the free group for generalised words; thus we were able to do all of the hard work in our specially designed free group, entirely with our specially designed restricted verbal subgroups.

By a result of Hall [8] every finitely generated virtually nilpotent group is verbally-concise, and hence verbally-robust. So combining this last result with Proposition 4.2.7:

**Corollary 4.2.16.** *Finitely generated virtually nilpotent groups are verbally-elliptic.*

### 4.3 Polycyclic pro- $p$ groups

Having introduced Roman'kov's theory of generalised words, we are now ready to return to the topic of profinite groups. Specifically, we will translate the results of the previous section to the profinite case, and combine them with the results of Chapter 3 to prove that a profinite group which is virtually a polycyclic pro- $p$  group is verbally-elliptic.

When dealing with profinite groups we will make a minor adjustment to our definition of a generalised word, and require that the automorphisms which appear in the generalised word are *topological* automorphisms. A finitely generated profinite group  $G$  is isomorphic to the inverse limit of its quotients  $G/N$ , where  $N$  runs over the open *topologically characteristic* subgroups of  $G$  (this follows from Proposition 1.6 of [2]). With our modification

in place, a generalised word  $w$  for  $G$  is also a generalised word for  $G/N$ . Therefore, by Proposition 1.2.4:

$$\|w(G)\| = \sup \|w(G/N)\|,$$

where  $N$  runs over the open topologically characteristic subgroups of  $G$ .

In the remark following Theorem 4.2.15 we noted that the width of a generalised word  $w$  in a  $d$ -generator nilpotent group of class  $c$  is bounded by a function of  $c$ ,  $d$ , and  $w$  only. Combining this with the above it follows that:

**Corollary 4.3.1.** *Every generalised word has finite width in a finitely generated nilpotent profinite group.*

Examining the proof of Proposition 4.2.7, we can see that the hypothesis that a word  $w$  be robust on every homomorphic image of  $E$  is required because we take the quotient of  $E$  by the verbal subgroup  $V(G)$ . The width of  $V(G)$  is finite, so if  $E$  and  $G$  are profinite groups then  $V(G)$  is closed. Therefore in this case we need only consider the *profinite* images of  $E$ . But every word is robust in a profinite group which is virtually a pro- $p$  group of finite rank by Lemma 1.3.4; therefore Proposition 4.2.7 becomes

**Proposition 4.3.2.** *Let  $G$  be a profinite group which is virtually a pro- $p$  group of finite rank. If  $G$  contains an open normal subgroup in which every generalised word has finite width then  $G$  is verbally-elliptic.*

In particular we have:

**Corollary 4.3.3.** *Every finitely generated nilpotent-by-finite profinite group which is virtually a pro- $p$  group is verbally-elliptic.*

This result follows immediately from Proposition 4.3.2, but in Section 4.4 we will show that every nilpotent-by-finite group is verbally-concise (Lemma 4.4.2). We can also combine Corollary 4.3.1 and Proposition 4.2.7 to achieve:

**Corollary 4.3.4.** *Every finitely generated nilpotent-by-finite profinite group is verbally-elliptic.*

Compare the proof of this next result to that of Proposition 4.2.7:

**Proposition 4.3.5.** *Suppose that a profinite group  $G$  which is virtually a pro- $p$  group of finite rank has the form  $G = NM$ , where  $N$  is a closed normal nilpotent subgroup and  $M$  is a closed nilpotent-by-finite group. Then  $G$  is verbally-elliptic.*

*Proof.* Let  $w(x_1, \dots, x_n)$  be any word. Consider the generalised words (for  $N$ ) which have the form

$$v(x_1, \dots, x_n) = w(m_{i_1}x_1, \dots, m_{i_n}x_n)w(m_{i_1}, \dots, m_{i_n})^{-1},$$

with  $m_{i_1}, \dots, m_{i_n} \in M$ . Let  $V$  be the set of all such generalised words and write  $K := V(N)$ . We will show that  $w(G/K)$  has finite width and that  $K \cap w(G)$  has finite width with respect to  $w\{G\}$ . We will start with the latter of these:

Every generalised word has finite width in  $N$ , so by Lemma 1.3.2  $K$  has finite width with respect to  $V\{N\}$ . Each element of  $V\{N\}$  is a product of

two elements of  $w\{G\}$ . Hence  $V(N)$  is contained in  $w(G)$  and has finite width with respect to  $w\{G\}$ . So  $\|K \cap w(G)\|_{w\{G\}}$  is finite.

We will now show that  $w(G/K)$  has finite width. Because  $K$  has finite width respect to  $w\{G\}$  it is a closed normal subgroup of  $G$ . The nature of the generalised words contained in  $V$  means that if we are working modulo  $K$  then

$$w(m_{i_1}g_1, \dots, m_{i_n}g_n) = w(m_{i_1}, \dots, m_{i_n})$$

for any  $g_1, \dots, g_n \in G$  and any  $m_i \in M$ . This means that there is a one-to-one correspondence between the  $w$ -values of  $G/K$  and the  $w$ -values of  $MK/K$ , and that this correspondence extends to an isomorphism between  $w(G/K)$  and  $w(MK/K)$ .

Now  $MK/K$  is finitely generated and virtually nilpotent, so  $\|w(MK/K)\|$  is finite by Corollary 4.3.3. Hence  $\|w(G)\|$  is finite.  $\square$

We are almost ready to show that every polycyclic pro- $p$  group is verbally-elliptic. Before we get to the main argument we will need just a little more technical preparation.

**Lemma 4.3.6.** *Let  $X$  and  $\Phi$  be sets. We will work in the group  $F(X \cup \Phi)$ .*

*Let  $w$  be a word in  $n$  variables. If  $x_1, \dots, x_n \in X$  and  $\phi_1, \dots, \phi_n \in \Phi$  then*

$$w(x_1\phi_1, \dots, x_n\phi_n) = w(\phi_1, \dots, \phi_n)w'(x_1, \dots, x_n), \quad (4.3)$$

*where  $w'$  is a product of words in  $x_1, \dots, x_n$  which have been conjugated by words in  $\phi_1, \dots, \phi_n$ . That is,  $w'(x_1, \dots, x_n)$  is a value of a generalised word in which the automorphisms are words in the elements of  $\Phi$ .*

Moreover, write  $w^{(l)}$  for an independent product of  $l$  copies of  $w$  or its inverse (so that  $w^{(l)}$  represents the symbolic expression of an element in a verbal subgroup of width  $l$ ). Then

$$w^{(l)}\{X\Phi\} = w^{(l)}\{\Phi\}v\{X\}, \quad (4.4)$$

where  $v$  is a product of  $l$  words in  $n$  variables which have been conjugated by words in some  $\phi_1, \dots, \phi_{ln} \in \Phi$ . That is,  $v(x_1, \dots, x_{ln})$  is a value of a generalised word in which the automorphisms are words in the elements of  $\Phi$ .

*Proof.* The proof of the first identity is a routine, but tedious, induction on the length of  $w$ , using the identity

$$x\phi = \phi x^\phi.$$

The second identity follows from the first in a similar way. We will not spell out the details here.  $\square$

The above result was effectively a result about words themselves, rather than their verbal subgroups in particular groups; therefore we can think of Equation (4.3) as an identity of words. We worked in the free group  $F(X \cup \Phi)$  because  $w$  was a normal (that is, not generalised) word, and could be viewed as an element of  $F(X)$ .

To do a similar thing with generalised words we will work in our free group for generalised words  $FG(X, \Phi)$ . Notice that a generalised word may be considered to be an element of this group.

**Lemma 4.3.7.** *Let  $X = \{x_1, \dots, x_n\}$ , let  $\Phi$  be any finite set, and write  $K := FG(X, \Phi)$ . Now let  $w(x_1, \dots, x_n)$  be any generalised word taking fixed values in  $\Phi$ . Fix  $m \in \mathbb{N}$  and define  $v(x_1, \dots, x_n) := w(x_1^m, \dots, x_n^m)$ . Write  $q(c) := m^{c(c+1)/2}$ ; then for each  $c \geq 0$ :*

$$w^{q(c)} \in v_r(K)\gamma_{c+1}(K),$$

where  $v_r(K)$  is the subgroup generated by the elements of the form  $v(k_1, \dots, k_n)$ , for  $k_1, \dots, k_n \in F(X)$ .

*Proof.* We will prove this by induction on  $c$ , the case  $c = 0$  being trivial. Suppose that  $c > 0$  and that the result is true for lower values of  $c$ . By hypothesis

$$w^{q(c-1)} = ay, \tag{4.5}$$

where  $a \in v_r(K)$  and  $y \in \gamma_c(K)$ . Then

$$w^{q(c)} = (w^{q(c-1)})^{m^c} = (ay)^{m^c} = a^{m^c} y^{m^c} z_1, \tag{4.6}$$

where  $z_1 \in \gamma_{c+1}(K)$ .

Now let  $\xi$  be the endomorphism of  $K$  which takes each  $x \in X$  to the element  $x^m$  and fixes the elements of  $\Phi$  (so  $\xi(w) = v$ ). From Equation (4.5) we thus have

$$\xi(y) = \xi(a)^{-1} \xi(w)^{q(c-1)} = \xi(a)^{-1} v^{q(c-1)}. \tag{4.7}$$

By the arguments stated in the proof of Lemma 4.2.9 it also follows that

$\xi(y) = y^{m^c} z_2$  for some  $z_2 \in \gamma_{c+1}(K)$ . Together with Equation (4.7), this gives

$$y^{m^c} = \xi(a)^{-1} v^{q(c-1)} z_2^{-1}.$$

Substituting this into Equation (4.6) then gives

$$w^{q(c)} = a^{m^c} \xi(a)^{-1} v^{q(c-1)} z_2^{-1} z_1.$$

Because  $\xi(v_r(K)) \leq v_r(K)$ , this is an element of  $v_r(K)\gamma_{c+1}(K)$ , as required.  $\square$

**Remark.** Notice in the above proof that  $w$  was actually treated as an element of  $FG(X, \Phi)$ . On the other hand,  $v$  was treated both as an element of  $FG(X, \Phi)$  (when we viewed it as the image of  $w$  under  $\xi$ ) and as a generalised word—producing a restricted vocabulary in  $K$ . Moreover, in the last line of the proof, we use the fact that the *element*  $v \in FG(X, \Phi)$  is actually a member of the *verbal subgroup*  $v_r(K)$ .

**Corollary 4.3.8.** *Let  $G$  be a finitely generated nilpotent profinite group and let  $w(x_1, \dots, x_n)$  be a generalised word for  $G$ . Fix  $m \in \mathbb{N}$  and define  $v(x_1, \dots, x_n) := w(x_1^m, \dots, x_n^m)$ . Then  $|w(G) : v(G)|$  is finite.*

*Proof.* Let  $V = v(G)$  and  $W = w(G)$ . By Corollary 4.3.1 both  $V$  and  $W$  are closed subgroups of  $G$ . Suppose that  $G$  has nilpotency class  $c$  and let  $q := q(c)$  be as given in Lemma 4.3.7. Then for all  $g_1, \dots, g_n \in G$  we have

$$w(g_1, \dots, g_n)^q \in V;$$

therefore  $\overline{W^q} \leq V$ . But  $W/\overline{W^q}$  is a finitely generated nilpotent profinite group of finite exponent; therefore  $W/\overline{W^q}$  is finite. Hence  $V$  has finite index in  $W$ .  $\square$

**Theorem 4.3.9.** *Every word has finite width in a profinite group which is virtually a polycyclic pro- $p$  group.*

*Proof.* Let  $G$  be a profinite group which is virtually a polycyclic pro- $p$  group and let  $w(x_1, \dots, x_n)$  be any word. If  $\tau(G)$  denotes the maximal finite normal subgroup of  $G$  then  $w$  has finite width in  $G$  if and only if  $w$  has finite width in  $G/\tau(G)$ . Therefore we may assume that the Fitting Subgroup  $N$  of  $G$  is torsion-free. Hence, by Theorem 3.2.1,  $G$  may be embedded as a subgroup of finite index in a profinite group  $G_1$  which is virtually a polycyclic pro- $p$  group and which has the form

$$G_1 = N_1G = N_1M_1,$$

where  $N_1$  is a closed normal nilpotent subgroup of  $G_1$  and a finite extension of  $N$ , and  $M_1$  is a closed virtually nilpotent subgroup. By Corollary 4.3.1  $w(G_1)$  has finite width  $l$ , say.

We can write every element of  $G_1$  in the form  $ag$ , with  $a \in N_1$  and  $g \in G$ . Given  $a_1, \dots, a_n \in N_1$  and  $g_1, \dots, g_n \in G$ , Equation (4.3) (with  $\Phi$  replaced by  $G$  and  $X$  replaced by  $N_1$ ) shows that a typical element of  $w\{G_1\}$  may be written as

$$w(a_1g_1, \dots, a_ng_n) = w(g_1, \dots, g_n)w'(a_1, \dots, a_n),$$

where  $w'$  is a value in  $N_1$  of a generalised word which takes fixed values in  $G$ . Note that if  $a_1, \dots, a_n$  are elements of  $N$  then  $w'$  has width at most 2 with respect to  $w\{G\}$ .

Each  $g \in w(G)$  has width no greater than  $l$  with respect to  $w\{G_1\}$ ; therefore we can use Equation (4.4) to write  $g$  in the form

$$g = y_g z_g, \quad (4.8)$$

where  $y_g$  has width no greater than  $l$  with respect to  $w\{G\}$ , and  $z_g$  is a value in  $N_1$  of a generalised word  $v_g(x_1, \dots, x_{ln})$  which takes fixed values in  $G$ . By examining the formulation of Equation (4.4) we can see that the elements of  $v_g(N)$  have width at most  $2l$  with respect to  $w\{G\}$ . (Note, however, that it is not necessarily the case that  $v_g(N_1) \leq w(G)$ .)

From Equation (4.8) we can see that  $w(G)$  will have finite width provided that the elements  $z_g$ , obtained by varying  $g$  over all elements of  $w(G)$ , have bounded width with respect to  $w\{G\}$ . In fact, we claim that the subgroup  $Z$  generated (abstractly) by these elements has finite width with respect to  $w\{G\}$ .

Suppose that  $a_1, \dots, a_d \in Z$  is a topological generating set for the closure of  $Z$ . Only finitely many  $v_g$  are required to obtain these elements; suppose that  $t$  such expressions are required. As always we will replace this finite collection with a single generalised word  $v(x_1, \dots, x_{lnt})$ . By Corollary 4.3.1  $v(N_1)$  is closed, so we have

$$\overline{\langle a_1, \dots, a_d \rangle} \leq v(N_1);$$

that is  $Z \leq v(N_1)$ . Now let  $m := |N_1 : N|$  and define

$$u(x_1, \dots, x_{nlt}) := v(x_1^m, \dots, x_{nlt}^m).$$

The elements of  $u\{N_1\}$  are elements of  $v\{N\}$ , so they have width at most  $2lt$  with respect to  $w\{G\}$ . Additionally, the verbal subgroup  $u(N_1)$  has finite width. Therefore  $u(N_1)$  has finite width with respect to  $w\{G\}$ .

Finally, consider  $Z_1 = \langle Z, u(N_1) \rangle$ . We have  $Z_1 \leq w(G)$  and

$$u(N_1) \leq Z_1 \leq v(N_1).$$

The index  $|v(N_1) : u(N_1)|$  is finite by Corollary 4.3.8, and so  $|Z_1 : u(N_1)|$  is also finite. It follows that  $Z_1$  has finite width with respect to  $w\{G\}$ . Certainly then  $Z$  has finite width with respect to  $w\{G\}$ , and the proof is complete.  $\square$

Combining this with Lemma 4.1.1 we arrive at our main result.

**Theorem 4.3.10.** *Every profinite group which is virtually a pro- $p$  group of finite rank is verbally-elliptic.*

## 4.4 A note on finite rank nilpotent groups

In Section 4.2 we considered the verbal subgroups of finitely generated nilpotent groups and related groups; in that section the hypothesis that the groups be finitely generated was integral to the proofs we gave. We will now illustrate a simple technique which can be used to extend these results to a weaker hypothesis.

There are a couple of things that the reader should note before we proceed. Firstly, we are not going to be working with profinite groups here; the material, however, is strongly related to the topics of this chapter. Secondly, in this section we will revert to the more common definition of finite rank, and say that a group has rank  $d$  if every finitely generated subgroup can be generated by at most  $d$  elements.

**Proposition 4.4.1.** *Every generalised word has finite width in a nilpotent group of finite rank.*

*Proof.* Let  $G$  be a nilpotent group of class  $c$  and rank  $d$ , and let  $w(x_1, \dots, x_n)$  be a generalised word. Take any  $h \in w(G)$  and suppose that

$$h = w(\mathbf{g}_1) \cdots w(\mathbf{g}_s),$$

where  $\mathbf{g}_1, \dots, \mathbf{g}_s$  are  $n$ -tuples of elements of  $G$ . Let  $H$  be the subgroup generated by the elements of  $G$  which appear in these  $n$ -tuples. So  $h \in w(H)$ , and

$$\|h\|_{w\{G\}} \leq \|w(H)\|.$$

But  $H$  is a  $d$ -generator nilpotent group of class at most  $c$ , so by Theorem 4.2.15 the width of  $w(H)$  is bounded by a function depending only on  $c$ ,  $d$ , and  $w$ . Since  $h$  was arbitrary, it follows that the width of  $w(G)$  is also bounded by this function.  $\square$

We can apply this idea further: If  $G$  is a virtually nilpotent group and  $w\{G\}$  is finite then  $G$  contains a finitely generated subgroup  $H$  such that

$w\{H\} = w\{G\}$ . Because finitely generated virtually nilpotent groups are verbally-concise,  $w(H)$  is finite. This leads to the following result:

**Lemma 4.4.2.** *Every virtually nilpotent group is verbally-concise.*

Hence by Proposition 4.2.7:

**Proposition 4.4.3.** *Every virtually nilpotent group of finite rank is verbally-elliptic.*

The arguments of Proposition 4.3.5 now apply (and we need only remove the various instances of the word *closed* from the proof), to give:

**Theorem 4.4.4.** *Let  $G$  be group of finite rank which can be written in the form  $G = NM$  with  $N$  a normal nilpotent subgroup and  $M$  a virtually nilpotent subgroup. Then  $G$  is verbally-elliptic.*

It is possible that this could be a first step in a proof that soluble minimax groups are verbally elliptic, for such a group can be embedded in a group of the above form. For the most part, the strategy of Theorem 4.3.9 generalises to the case of soluble minimax groups: The initial part of the proof relies only on the fact that a group of the form  $NM$  (with  $N$  and  $M$  as above) is verbally-elliptic, together with some identities of words. In the last part of the proof the ‘action’ takes place within  $N$  (and suitable finite extensions of  $N$ ), so by the results of this section these arguments will also generalise to the soluble minimax case. The problem, however, is with the short step which bridges these two parts together. Namely, we consider the subgroup  $Z$  generated by the ‘error terms’  $z_g$ . In our original proof we know that  $Z$  is finitely generated; so it can be constructed from only finitely many

generalised words  $v_g$ . If  $G$  is a soluble minimax group then this need not be the case. Worse still, because the last part of the proof only tells us that the width of  $Z$  with respect to  $w\{G\}$  is *finite*, and does not provide a bound independent of the elements of  $Z$ , we cannot employ the techniques of this section to get around this problem.

Despite this, it is quite reasonable to expect that, with a new idea, a proof along these lines could be made to work. It is also possible that the proof of Theorem 4.3.9 relies so inherently on every subgroup being finitely generated that a completely new strategy is required. Of course, it may be that soluble minimax groups are not verbally-elliptic, but I expect that my inability to find a proof is due to lack of skill, and not that the result is false. Hence I make the following conjecture:

**Conjecture 2.** *Every soluble minimax group is verbally-elliptic.*

## 4.5 Notes

The vast majority of the material in Section 4.2 is due to Roman'kov, although the exposition here is quite different to Roman'kov's original presentation of the material in [29] (and far more detailed). The main strategies behind the proof of Proposition 4.3.5 and Theorem 4.3.9 are also due to Roman'kov, but again their formulation here is much more explicit.

As commented in the notes of Chapter 3, we can prove that a virtually polycyclic group with torsion-free Fitting subgroup can be embedded in a group as a subgroup of finite index in a group  $G_1$  of the form  $G_1 = N_1G = N_1M_1$ , with  $N_1$  a finitely generated normal nilpotent subgroup and  $M_1$  a

finitely generated virtually nilpotent subgroup. Therefore the arguments of Theorem 4.3.9 will apply (once we remove all references to the topology) to give the following extension of Roman'kov's original result:

**Theorem.** *Every polycyclic-by-finite group is verbally-elliptic.*

A quick note about our proof of Theorem 4.3.9: We start by factoring out the maximal finite normal subgroup of  $G$  in order to ensure that the Fitting subgroup of  $G$  is torsion-free. The reader may wonder whether it would be sufficient to simply factor out the torsion part of the Fitting subgroup; the answer is no. For a counterexample consider the group  $G = C_\infty \times S_3$ . This group has Fitting subgroup  $C_\infty \times A_3$  which has torsion part isomorphic to  $A_3$ . The quotient of  $G$  by this group is isomorphic to  $C_\infty \times C_2$  which is equal to its own Fitting subgroup, but not torsion-free. Of course we could iterate this process to achieve the desired effect, but this author prefers to take the quotient by the maximal finite normal subgroup.

On numerous occasions in this chapter we have used the fact that there is a bound  $g(c, d, w)$  for the width of a word  $w$  in a  $d$ -generator nilpotent group of class  $c$ . It is worth remarking that the proofs of Corollary 4.3.1 and Theorem 4.4.1 show that  $g$  is also a bound for the width of  $w$  in a nilpotent group of class  $c$  and rank  $d$  (whether profinite or otherwise). In the non-profinite case this is a somewhat surprising result, for we might have expected that the width would be greater, given that the group itself need not be finitely generated. In fact, suppose that  $\mathcal{P}$  is a property of groups such that every subgroup of a group with  $\mathcal{P}$  also has  $\mathcal{P}$ . If the width of a word  $w$  in a

$d$ -generator  $\mathcal{P}$ -group is bounded by a function of  $d$ , then the width of  $w$  in a  $\mathcal{P}$ -group with rank  $d$  is also bounded by the same function of  $d$ . Theorem 4.2.15 guarantees us that such a bound  $g(c, d, w)$  does exist for  $d$ -generator nilpotent groups of class  $c$ ; unfortunately it provides us with no idea as to what this bound might be.

The reader may have noticed that the ‘free group for generalised words’  $FG(X, \Phi)$  introduced for the proof of 4.2.15 is also used as a ‘group of generalised words’ in Lemma 4.3.7. This is an exact analogy of the fact that normal words may be viewed as elements of the free group  $F(X)$ . This leads to an interesting question: Should we view results such as Theorem 4.2.15 as results about verbal subgroups, or as results about generalised words themselves and how they can be structurally manipulated?

This question is easier to think about in the context of normal words: Firstly we could take the opinion that a product of a word with itself an arbitrary number of times has a certain structural property which enables it to be rewritten as a product of some fixed number  $\|w\|$  of words, together with an ‘error term’ whose values will always lie in the  $(c + 1)$ th term of the lower central series of a group. Alternatively we could take the opinion that the verbal subgroups of a finitely generated free nilpotent group have finite width, and that every verbal subgroup of a finitely generated nilpotent group is the image of such a subgroup.

After careful consideration I have decided that, in practice, the differences here are little more than semantic. Provided that we are aware of the close interplay between ‘structural identities’ and the properties of a word in a free

group, we are free to choose whichever approach makes a result more illuminating or more accessible. For Theorem 4.2.15 I believe that the ‘properties of the free group’ approach is more illuminating (because it just deals with verbal subgroups), while for Lemma 4.3.7 I believe that the ‘structural identity’ approach is more illuminating (because it involves altering the variables in a generalised word).

Finally, note that although we have used generalised words quite a lot in this chapter, we have never had to use any properties of any specific automorphisms. That is, our results have been true simply because of the nature of generalised words themselves. Given this, the reader may well ask whether the ‘symbolic’ approach to generalised words is the better approach. I would argue that because we have been specifically studying groups in which all generalised words have finite width, it is not surprising that we have not had to use any properties of specific automorphisms. Nothing we have seen suggests that there are not groups in which, for example, a generalised word has finite width provided it contains only automorphisms of order 2 or less.

Furthermore, even if we do take the view that the generalised words are the elements of  $FG(X, \Phi)$ , and that a specific choice of automorphisms gives an ‘instance’ of a generalised word, then each element of  $FG(X, \Phi)$  can only represent one ‘instance’ at a time. For example, in the proof of Theorem 4.3.9 the generalised words  $v_g$  are symbolically the same generalised word, but with different elements of  $G$  for automorphisms. Part of the difficulty of this proof (and the reason that the proof does not immediately generalise to soluble minimax groups) is dealing with the fact that each  $v_g$  behaves as a separate generalised word. Given this, why not just consider generalised

words to be different if they contain different automorphisms? This does not stop us recognising that different generalised words may be symbolically the same.

My personal opinion is that to define generalised words by Definition 4.2.1, and then view the elements of  $FG(X, \Phi)$  as ‘symbolic representations’ of these words is more useful to anyone who wishes to apply the theory of generalised words, and more accessible to people who are unfamiliar with the concept.

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